Continuity criteria

KET constructs $\{X_t : t \in T\}$ on $(\mathcal{X}^T, \Sigma^{\otimes T})$

Q: When can we ensure continuity? (T = metric space)

A: No way (when T uncountable)

$U = \mathrm{Exp}(1) \perp\!\!\!\perp X$ $\quad (\mathcal{X} = \mathbb{R}, T = [0,\infty))$

$$Y_t := \begin{cases} X_t + 1 & t = U \\ X_t & t \neq U \end{cases}$$

$\forall t \in [0,\infty)$: $X_t = Y_t$ a.s. (t)

so X and Y have same FDD's.

A deeper issue: $X_t(\omega) := \omega_t$, $\mathcal{X} = \mathbb{R}$, $\Sigma = \mathcal{B}(\mathbb{R})$

$\{t \longmapsto X_t \text{ cont.}\} \notin \mathcal{B}(\mathbb{R})^{\otimes [0,\infty)}$ (Folland)

Clearly we need more work.

2 alternatives:
- define extension to a larger σ-alg. $(\mathcal{B}(\mathbb{R}^{[0,\infty)}))$
- work with modifications

<u>Def</u> $Y = \{Y_t : t \in T\}$ is a <u>version</u>/<u>modification</u> of $X = \{X_t : t \in T\}$ if $\forall t \in T$: $P(X_t = Y_t) = 1$

Q: Criteria for existence of continuous version?

Let's try: Necessary cond:

Def A process $\{X_t : t \in T\}$ taking values in metric space $(\mathcal{X}, \rho)$ is said **stochastically continuous** if

$$\forall t \in T \ \forall \varepsilon > 0: \ \lim_{s \to t} P\big(\rho(X_t, X_s) > \varepsilon\big) = 0.$$

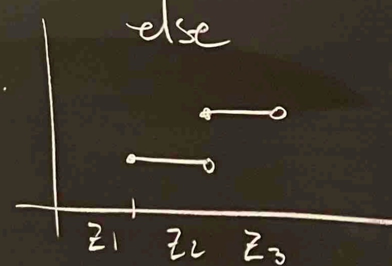
Not sufficient !

Ex Let $Z_1, Z_2, \ldots$ be iid $\mathrm{Exp}(1)$. Set

$$N_t := \begin{cases} \sup\{m \geq 0 : \sum_{i=1}^{m} Z_i \leq t\} & \text{on } \{\sum_{i=1}^{\infty} Z_i = \infty\} \\ 0 & \text{else} \end{cases}$$

Then $\{N_t : t \geq 0\}$ is PP we defined earlier.

The paths are RCLL yet not continuous.



$$P(|N_t - N_s| \geq \varepsilon) \underset{\varepsilon \in (0,1)}{=} P(|N_t - N_s| \geq 1) = 1 - e^{-|t-s|} \xrightarrow[s \to t]{} 0$$

so is stoch. continuous!

$E(|N_t - N_s|^a) \geq P(|N_t - N_s| \geq 1) = 1 - e^{-|t-s|} \approx |t-s|$

So linear asymptotic in $|t-s|$ not enough.

"Anything" better will do!

Thm (Kolmogorov-Čentsov) Let $\{X_t : t \geq 0\}$ be $\mathbb{R}$-valued s.t.

$$\exists C, a, b > 0 \ \forall t, s \geq 0: \ E(|X_t - X_s|^a) \leq C|t-s|^{1+b}$$

Then X admits a continuous version $\tilde{X}$ s.t.

$$\forall \gamma \in (0, \tfrac{b}{a}) \ \forall t_0 > 0: \ \sup_{0 \leq s < t \leq t_0} \frac{|\tilde{X}_t - \tilde{X}_s|}{|t-s|^\gamma} < \infty \ \text{a.s.}$$

So we have locally γ-Hölder for all $\gamma < b/a$

Pf: Markov ineq: $P(|X_t - X_s| > u) \leq \frac{1}{u^a} E(|X_t - X_s|^a) \leq \frac{C}{u^a} |t-s|^{1+b}$

Let $\gamma \in (0, b/a)$. Then

$$\forall t \geq 0 \ \forall n \geq 1: \ P(|X_{t+2^{-n}} - X_t| > 2^{-n\gamma}) \leq C 2^{-n(1+b)+n\gamma a} = C 2^{-n} 2^{-n\underbrace{(b-\gamma a)}_{>0}}$$

Hence $\forall L \geq 1$:

$$P\left(\max_{k=1,\dots,L2^n} |X_{k2^{-n}} - X_{(k-1)2^{-n}}| > 2^{-n\gamma}\right) \leq CL 2^{-n(b-\gamma a)}$$

By Borel-Cantelli: $\exists\, n_0 = n_0(\omega)$ with $P(n_0 < \infty) = 1$
s.t. $\forall n \geq n_0(\omega)\ \forall k = 1, \dots, L2^n$: $|X_{k2^{-n}} - X_{(k-1)2^{-n}}| \leq 2^{-n\gamma}$

Denote $\mathbb{D}_n := \{ k2^{-n} : k = 0, \dots, L2^{+n} \}$, $\mathbb{D}_{n-1} \subseteq \mathbb{D}_n$

<u>Note</u> if $s,t \in \mathbb{D}_n$, $s < t$, then there exist $s', t' \in \mathbb{D}_{n-1}$ s.t. $s \leq s' \leq t' \leq t$ and $s - s' \leq 2^{-n}$, $t - t' \leq 2^{-n}$

Then $|X_t - X_s| \leq |X_t - X_{t'}| + |X_s - X_{s'}| + |X_{t'} - X_{s'}|$

So
$$\max_{\substack{t,s \in \mathbb{D}_n \\ |t-s|<\delta}} |X_t - X_s| \underset{n \geq n_0}{\leq} 2 \cdot 2^{-n\gamma} + \underbrace{\max_{\substack{t',s' \in \mathbb{D}_{n-1} \\ |t'-s'|<\delta}} |X_{t'} - X_{s'}|}_{= 0 \text{ if } 2^{-(n-1)} \leq \delta}$$

Hence we get
$$\sup_{\substack{t,s \in \bigcup_{n > n_0} D_n \\ |t-s|<\delta}} |X_t - X_s| \leq \sum_{n : 2^{-n} \leq \delta} 2 \cdot 2^{-\gamma n} = \frac{2}{1 - 2^{-\gamma}} \delta^{\gamma}$$

So X is uniformly γ-Hölder on dyadic points

[3] AH lemma:

$$\tilde{X}_t := \begin{cases} \lim\limits_{\substack{s \to t \\ s \in \bigcup_{n > n_0} D_n}} X_s & \text{on } \{ n_0 < \infty \text{ for all } L \geq 1 \} \\ X_0 & \text{else} \end{cases}$$

(the limit exists by γ-Hölder property)

Then $\tilde{X}$ is locally γ-Hölder. Since

$$P(|X_t - X_s| > \varepsilon) \leq \frac{1}{\varepsilon^a} E(|X_t - X_s|^a) \leq \frac{C}{\varepsilon^a} |t-s|^{1+b}$$

By Fatou, the bound gives

$$\forall \varepsilon > 0 : P(|\tilde{X}_t - X_t| > \varepsilon) = 0. \Rightarrow \tilde{X} \text{ is a version of } X.$$

Taking $\gamma \uparrow b/a$ we cover them all. $\boxtimes$

Thm A standard Brownian motion exists and has locally γ-Hölder path for all $\gamma < 1/2$

Pf: KET gives $\{ B_t : t \geq 0 \}$ with correct FDD's.

$$E(|B_t - B_s|^a) = E(|N(0, |t-s|)|^a) = |t-s|^{a/2} E(|N(0,1)|^a)$$

$b = \frac{a}{2} - 1 \quad (a > 2)$

$\gamma < \frac{b}{a} = \frac{a/2 - 1}{a} \xrightarrow[a \to \infty]{} \frac{1}{2}.$

KC gives γ-Hölder version $\forall \gamma < 1/2$. $\boxtimes$

Q: (Next time) In what sense is SBM unique?

Remark For $\{X_t : t \in \mathbb{R}^d\}$ the KC condition

becomes $\forall t, s \in \mathbb{R}^d$: $E\left(|X_t - X_s|^a\right) \leq C \|t - s\|^{d+b}$

dimension