Girsanov & Novikov

<u>Last time:</u> <u>Girsanov's thm:</u> Let $Y \in \mathcal{V}_B^{loc}$, $M_t := \exp\left\{ \int_0^t Y_s \, dB_s - \frac{1}{2}\int_0^t Y_s^2 \, ds \right\}$

Then $M \in \mathcal{M}_{loc}^{cont}$. If $EM_t = 1$, then

$\left\{ \tilde{B}_s := B_s - \int_0^s Y_u \, du \right\}_{s\in[0,t]}$ is SBM under $\tilde{P}(A) := E(\mathbb{1}_A M_t)$.

Condition $EM_t = 1$ is both subtle & non-trivial

<u>Recall</u> d-dim. Bessel process solves $dX_t = \frac{d-1}{2X_t} dt + dB_t$, $X_t = 1$

For $d > 2$: $\inf_{t \geq 0} X_t > 0$ a.s.

<u>Lemma</u> Let $d > 2$, X = d-dim Bessel process, $X_0 = 1$

Set

$$M_t = X_t^{2-d} = \exp\left\{ (2-d)(\log X_t - \log X_0) \right\}$$

$$= \exp\left\{ (2-d)\int_0^t \frac{1}{X_s} dB_s - \frac{(d-2)^2}{2}\int_0^t \frac{1}{X_s^2} ds \right\}$$

Then $M \in \mathcal{M}_{loc}^{cont}$ yet $\forall t > 0$: $EM_t < 1$.

Pf: Formula proved by applying Itô to $\log X_t$.
Suppose $\exists T>0$ s.t. $E M_T = 1$. Set $\tilde{P}(A) := E(\mathbb{1}_A M_T)$.
Girsanov: $\tilde{B}_t := B_t + (d-2)\int_0^t \frac{1}{X_s}\,ds$ is SBM (for $t\in[0,T]$)

Then $dX_t = \frac{d-1}{2X_t}dt + dB_t = \underbrace{\left(\frac{d-1}{2X_t} - \frac{d-2}{X_t}\right)}_{= \frac{3-d}{2X_t} = \frac{d'-1}{2X_t}} dt + d\tilde{B}_t$

Set $d'-1 := 3-d$ i.e. $d' = 4-d$
For $d>2$ we have $d'<2$. So

$$\tilde{P}\left(\inf_{t\in[0,T]} X_t = 0\right) > 0 \text{ yet } P\left(\inf_{t\in[0,T]} X = 0\right) = 0$$

! with $\tilde{P} \ll P$.

$\boxtimes$

Remark: For this M we have $\sup_{t\geq 1} E M_t^p < \infty$ $\forall p\in[0,\frac{d}{d-2}]$, $\forall t>0$.
So M is UI loc. martingale which is NOT a martingale !.
Indeed martingale property needs UI of $\{M_{t\wedge\tau_n} : n\geq 1\}$.
stopping times making M martingale

Q: What conditions ensure $EM_t = 1$?

Lemma Let $X \in \mathcal{M}^{cont}_{loc}$, $M_t := \exp\{X_t - \frac{1}{2}\langle X\rangle_t\}$.

Then $\forall t \geq 0$:

$$\exists \varepsilon > 0 : E\left(e^{(\frac{1}{2}+\varepsilon)\langle X\rangle_t}\right) < \infty \Rightarrow E M_t = 1$$

Pf: Note $M \in \mathcal{M}^{cont}_{loc}$ because Itô: $dM_t = M_t(dX_t - \frac{1}{2}d\langle X\rangle_t) + \frac{1}{2}M_t d\langle X\rangle_t$
$= M_t dX_t$.

$\tau_n := \inf\{t \geq 0 : X_t \geq n\}$

Then $M_{t\wedge\tau_n} \leq e^n$ so $\{M_{t\wedge\tau_n} : t \geq 0\} \in \mathcal{M}^{cont}$

Pick $\lambda > 1$, $p > 1$, set q s.t. $\frac{1}{p} + \frac{1}{q} = 1$.

$$E(M^{\lambda}_{t\wedge\tau_n}) = E\left(\exp\left\{\lambda X_{t\wedge\tau_n} - \frac{\lambda}{2}\langle X\rangle_{t\wedge\tau_n}\right\}\right)$$

$$\overset{\text{arrange for Hölder}}{=} E\left(\exp\left\{\lambda X_{t\wedge\tau_n} - \frac{\lambda^2 p}{2}\langle X\rangle_{t\wedge\tau_n}\right\}\exp\left\{\frac{\lambda^2 p - \lambda}{2}\langle X\rangle_{t\wedge\tau_n}\right\}\right)$$

$$\leq \underbrace{E\left(\exp\left\{\lambda p X_{t\wedge\tau_n} - \frac{\lambda^2 p^2}{2}\langle X\rangle_{t\wedge\tau_n}\right\}\right)^{1/p}}_{=1} E\left(\exp\left\{\frac{\lambda^2 p - \lambda}{2} q \langle X\rangle_{t\wedge\tau_n}\right\}\right)^{1/q}$$

$$\sup_{n\geq 1} E(M^{\lambda}_{t\wedge\tau_n}) \leq \left[E\left(\exp\left\{\frac{\lambda^2 p - \lambda}{2} q \langle X\rangle_t\right\}\right)\right]^{1/q}$$

Now $\frac{\lambda^2 p - \lambda}{2} q \underset{q=\frac{p}{p-1}}{=} \frac{1}{2}\frac{\lambda p - 1}{p-1}\lambda p \xrightarrow[\substack{\lambda \downarrow 1 \\ p \downarrow 1}]{} \frac{1}{2}$

So we can choose $\lambda > 1$, $p > 1$ s.t. $\leq \frac{1}{2} + \varepsilon$.

Then $\{M_{t\wedge\tau_n}\}_{n\geq 1}$ is UI and so $1 = E(M_{t\wedge\tau_n}) \xrightarrow[n\to\infty]{} E(M_t)$. $\boxtimes$

Examples exists that shows $E\left(e^{(\frac{1}{2}-\varepsilon)\langle M\rangle_t}\right) < \infty$ is NOT sufficient for $EM_t = 1$, for any $\varepsilon > 0$.

Thm (Novikov 1972) Let $X \in \mathcal{M}^{\text{cont}}_{\text{loc}}$, $M_t := \exp\{X_t - \frac{1}{2}\langle X\rangle_t\}$. Then $\forall t \geq 0$: $E\left(e^{\frac{1}{2}\langle M\rangle_t}\right) < \infty \Rightarrow EM_t = 1$.

Pf: See textbooks.

Applications of Girsanov

— solve SDE $\quad dX_t = a(t, X_t)\,dt + dB_t$

"solved" by change of measure $\quad \widetilde{P}_t(A) = E^x\left(1_A \exp\left\{ \int_0^t a(s, X_s)\, \underbrace{dX_s}_{\text{SBM}} - \frac{1}{2}\int_0^t a(s,X_s)^2\, ds \right\}\right)$
$A \in \mathcal{F}_t^X.$

Assuming $\forall t \ge 0: \; E^x\left(\exp\left\{ \frac{1}{2}\int_0^t a(s,X_s)^2\, ds \right\}\right) < \infty.$

Then $\{\widetilde{P}_t\}_{t\ge 0}$ for a consistent family on $\bigcup_{t\ge 0} \mathcal{F}_t^X$

By Kolmogorov Ext. Thm $\Rightarrow$ extend to a $\widetilde{P}$ prob. on $\mathcal{F}_\infty := \sigma\left(\bigcup_{t\ge 0} \mathcal{F}_t^X\right)$

Under $\widetilde{P}$, $\quad B_t := X_t - \int_0^t a(s, X_s)\,ds$ is SBM $\Rightarrow dX_t = a(t, X_t)\,dt + dB_t.$

— Brownian motion conditioned to be > 0.

Thm Let P^x = law of 1-dim. SBM $\{B_t : t \ge 0\}$ started for $x > 0$.

Set $\tau_0 := \inf\{t \ge 0 : B_t = 0\}$. For $t > 0$ set

$$Q_t^x(A) := P^x(A \mid \tau_0 > t), \quad A \in \mathcal{F}_t.$$

Then $\forall s \ge 0 \; \forall A \in \mathcal{F}_s$: $\; Q^x(A) := \lim_{t\to\infty} Q_t^x(A)$ exists, and extends to prob. measure on $\mathcal{F}_\infty = \sigma(\bigcup_{t\ge 0} \mathcal{F}_t)$. Moreover,

B under $Q^x \overset{\text{law}}{=}$ 3-dimensional Bessel process

Pf $h(t,x) := P^x(\tau_0 > t)$. Then

Doob h-transform.

$$Q_T^x(A) \underset{A \in \mathcal{F}_t}{:=} E^x\left(\mathbb{1}_{A \cap \{\tau_0 > t\}} \frac{h(T-t, B_t)}{h(T, B_0)}\right)$$

Fact (by Reflection principle) $\lim_{t \to \infty} \sqrt{t}\, h(t,x) = \sqrt{\frac{2}{\pi}}\, x$

So $Q_T^x(A) \xrightarrow[T \to \infty]{} E^x\left(\mathbb{1}_A \frac{B_t}{B_0}\right)$

Now Itô calculus:

$$\frac{B_t}{B_0} = \exp\{\log B_t - \log B_0\}$$

$$= \exp\left\{\int_0^t \frac{1}{B_s}\, dB_s + \frac{1}{2}\int_0^t -\frac{1}{B_s^2}\, ds\right\}$$

Girsanov: Under Q^x: $\tilde{B}_t := B_t - \int \frac{1}{B_s} ds$ is SBM.

which means: $dB_t = \frac{1}{B_t} dt + d\tilde{B}_t$ ... B is 3-dim Bessel process. $\boxtimes$