Girsanov's Theorem

<u>idea</u> remove/modify drift term by "tilting" the measure

<u>Lemma</u> (Tilt of iid's) Let $X_1, \ldots, X_n$ be independent s.t. $\forall \lambda \in \mathbb{R}$: $\varphi_i(\lambda) := E(e^{\lambda X_i}) < \infty$. Define for $A \in \sigma(X_1, \ldots, X_n)$, $\lambda_1, \ldots, \lambda_n \in \mathbb{R}$

$$P_{\underline{\lambda}}(A) := E\left(1_A \exp\left\{ \sum_{i=1}^{n} [\lambda_i X_i - \log \varphi_i(\lambda_i)] \right\} \right)$$

Then $P_{\underline{\lambda}}$ is prob. measure s.t. $X_1 \ldots X_n$ are indep under $P_{\underline{\lambda}}$ with

$$E_{\underline{\lambda}}(X_i) = \frac{\varphi_i'(\lambda_i)}{\varphi_i(\lambda_i)} \qquad i = 1, \ldots, n.$$

<u>Pf</u> Setup $\Rightarrow P_{\vec{\lambda}}$ = probability

product form $\Rightarrow X_1 \ldots X_n$ are still indep under $P_{\underline{\lambda}}$.

$$E_{\lambda}(X_i) = E(X_i \exp\{\lambda_i X_i\}) / \varphi_i(\lambda_i) = \varphi_i'(\lambda_i)/\varphi_i(\lambda_i). \ \boxtimes$$

<u>Note</u> A key tool in Large-deviation theory (~~Sanov's~~ Cramér's theorem)

For Gaussians we can handle dependent r.v.:

Lemma Let $X = (X_1 \dots X_n) = N(0, C)$. For $\lambda = (\lambda_1 \dots \lambda_n) \in \mathbb{R}^n$

set $P_\lambda(A) = E\left(1_A \exp\left\{ \lambda \cdot X - \frac{1}{2} \lambda \cdot C\lambda \right\}\right)$

Then P_λ is a probability measure and

$$X - C\lambda \text{ under } P_\lambda \overset{\text{law}}{=} X \text{ under } P$$

$$\left(\text{i.e. } X = N(C\lambda, C) \text{ under } P_\lambda \right)$$

$E(e^{t\cdot X}) = e^{t\cdot EX + \frac{1}{2} t\cdot Ct}$

Pf: Let $t \in \mathbb{R}^n$. Then

$$E_\lambda\left(e^{t\cdot(X - C\lambda)}\right) = E\left(e^{(t+\lambda)\cdot X}\right) e^{-t\cdot C\lambda - \frac{1}{2}\lambda\cdot C\lambda}$$

$$= e^{\frac{1}{2}(t+\lambda)\cdot C(t+\lambda) - t\cdot C\lambda - \frac{1}{2}\lambda\cdot C\lambda}$$

Cramér-Wold

$$= e^{\frac{1}{2} t\cdot Ct} = E\left(e^{t\cdot N(0,C)}\right)$$

$\Rightarrow X - C\lambda$ under $P_\lambda = N(0, C)$. ☒

Thm (Girsanov 1960) Let $B = \mathrm{SBM}$ on $(\Omega, \mathcal{F}, P)$ adapted to $\{\mathcal{F}_t\}_{t\ge 0}$.

Let $Y \in \mathcal{V}^{loc}$ and for some $t > 0$, set,

$$M_t := \exp\left\{ \int_0^t Y_s \, dB_s - \frac{1}{2} \int_0^t Y_s^2 \, ds \right\}$$

Assume $EM_t = 1$ and, for $A \in \mathcal{F}_t$, set $\widetilde{P}(A) = E(\mathbb{1}_A M_t)$.

Then $\widetilde{P}$ is a prob. measure on $(\Omega, \mathcal{F}_t)$ and

$$\left\{ B_s - \int_0^s Y_u \, du \right\}_{s \in [0,t]} \text{ under } \widetilde{P} \overset{\text{law}}{=} \{ B_s : s \in [0,t] \} \text{ under } P$$

Lemma (Explanation of $EM_t = 1$) For above setting:

- $M \in \mathcal{M}^{cont}_{loc}$
- $EM_t = 1 \implies \{ M_{s \wedge t} : s \ge 0 \} \in \mathcal{M}^{cont}$

Pf: Itô formula: $dM_t = M_t \left(Y_t \, dB_t - \frac{1}{2} Y_t^2 \, dt \right) + \frac{1}{2} M_t Y_t^2 \, dt$

$= M_t Y_t \, dB_t$

Define $\tau_n := \inf\left\{ t \geq 0 : \left| \int_0^t Y_s \, dB_s \right| \geq n \right\}$.

Then $0 \leq M_{t \wedge \tau_n} \leq e^n$ so $\{ M_{t \wedge \tau_n} : t \geq 0 \}$ is bounded loc. martingale $\Rightarrow$ it's a martingale.

$Y \in \mathcal{V}^{loc} \Rightarrow \tau_n \xrightarrow[n \to \infty]{} \infty$ a.s. $\Rightarrow M \in \mathcal{M}^{cont}_{loc}$.

2nd part: Notice: $E(M_{t \wedge \tau_n} \mid \mathcal{F}_s) = M_{t \wedge s \wedge \tau_n}$ a.s.

So also $E(M_{t \wedge \tau_n}) = E(M_0) = 1$

Now $\tau_n \to \infty$ a.s. implies $M_{t \wedge \tau_n} \xrightarrow[n \to \infty]{} M_t$ a.s. so Fatou gives:

$\underbrace{E(M_t \mid \mathcal{F}_s) \leq M_{t \wedge s}}_{(*)}$ $\wedge$ $E(M_t) \leq 1$. So $E(M_t) = 1 \Rightarrow (*)$ must hold with $=$ a.s.

$\boxtimes$

Proof of Girsanov's thm: Assume $EM_t = 1$. Set, for $s \leq t$,

$$\tilde{B}_s := B_s - \int_0^s Y_u \, du, \quad N_s := \exp\left\{ i \int_0^s Z_s \, d\tilde{B}_u + \frac{1}{2} \int_0^s Z_u^2 \, du \right\}$$

for some $Z \in \mathcal{V}_0$.

claim $\{ M_{s \wedge t} N_{s \wedge t} : s \geq 0 \} \in \mathcal{M}^{loc}_{cont}$.

$$M_s N_s = \exp\left\{ \int_0^s (Y_s + iZ_s) \, dB_s - \frac{1}{2} \int_0^s (Y_s + iZ_s)^2 \, ds \right\}$$

We same calculation as before.

We can't reuse: $\tau_n := \inf\{t \geq 0 : \int_0^t Y_s \, dB_s \geq n\}$

because $Z \in \mathcal{V}_0 \Rightarrow N$ is bounded.

Notice that $EM_t = 1$ implied $M_{t\wedge\tau_n} \to M_t$ in L^1.

Since N is bdd, also $M_{t\wedge\tau_n} N_{t\wedge\tau_n} \xrightarrow[n\to\infty]{} M_t N_t$ in L^1

This implies:

$$1 = E(M_0 N_0) = E(M_t N_t) = \tilde{E}(N_t).$$

Now set $Z_s := \sum_{j=1}^{m} \lambda_j \mathbb{1}_{(t_{j-1}, t_j]}(s)$ $\quad 0 = t_0 < t_1 < \dots < t_m = t.$

Then $N_t = \exp\left\{ i \sum_{j=1}^{n} \lambda_j (\tilde{B}_{t_j} - \tilde{B}_{t_{j-1}}) + \frac{1}{2} \sum_{j=1}^{m} \lambda_j^2 (t_j - t_{j-1}) \right\}.$

So $1 = E(N_t)$ implies

$$E\left(\exp\left\{ i \sum_{j=1}^{n} \lambda_j (\tilde{B}_{t_j} - \tilde{B}_{t_{j-1}}) \right\} \right) = e^{-\frac{1}{2} \sum_{j=1}^{m} \lambda_j^2 (t_j - t_{j-1})}$$

So $\tilde{B}$ has independent increments with correct laws.

Since $\tilde{B}$ is cont $\Rightarrow$ it's a SBM. $\boxed{\times}$

Why do we care about this?

Cor Suppose $a: \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R}$ is Borel meas. s.t. for X being SBM with $X_0 = x$ under P^x, for some $t > 0$:

$$E M_t = 1 \quad \text{for} \quad M_t := \exp\left\{ \int_0^t a(s, X_s)\, dX_s - \frac{1}{2}\int_0^t a(s, X_s)^2\, ds \right\}.$$

Then for $\tilde{P}^x(A) := E^x(1_A M_t)$, $B_s := X_s - \int_0^t a(s, X_s)\, ds$ is SBM with $B_s = x$ under $\tilde{P}^x$. In particular, X solves $dX_u = a(u, X_u)\,du + dB_u$ up to time t under $\tilde{P}^x$.

Pf Apply Girsanov's thm.

Note This works (for a.e. path) with minial regularity assumption on $t, x \longmapsto a(t, x)$.