Methods for solving SDEs

Last time: reduction to ODE: $dX_t = a(t,X_t) + dB_t \Rightarrow$ $V_t := X_t - B_t$ solves $\frac{dV_t}{dt} = a(t, V_t + B_t)$

Another method – change of variables (time-homogeneous case)

$$dX_t = a(X_t)dt + \sigma(X_t)dB_t$$

Take $f \in C^2$: Then

$$df(X_t) = \Big[\underbrace{a(X_t)f'(X_t) + \tfrac{1}{2}\sigma(X_t)^2 f''(X_t)}_{=0} \Big] dt + f'(X_t)\sigma(X_t)dB_t$$

if we set $f(x) := \int_{x_0}^{x} \exp\left\{-2\int_{x_0}^{y} \frac{a(z)}{\sigma(z)^2} dz\right\} dy$

Thm Suppose that for some $\alpha < \beta$ $x \longmapsto \frac{a(x)}{\sigma^2(x)} \in L^{1,loc}((\alpha,\beta))$ and define f as above. Assume $f(x) \xrightarrow[x \downarrow \alpha]{} -\infty$, $f(x) \xrightarrow[x \uparrow \beta]{} +\infty$. Then X solves above SDE with initial data $X_0 = x_0$ if and only if $Z_t := f(X_t)$ solves SDE $dZ_t = \tilde{\sigma}(Z_t)dB_t$ for $\tilde{\sigma}(z) := \sigma \circ f^{-1}(z)\, f' \circ f^{-1}(z)$

In particular,

$$\forall t \geq 0: \ X_t \in (\alpha, \beta).$$

Pf Suppose X solve its SDE. Assumptions ensure: $f \in C^1((\alpha,\beta))$, $f' \in AC((\alpha,\beta))$
so Itô formula can be used. Differentiating shows Z stops its SDE.
To show X never leaves (α,β) we need:

Lemma Assume Z solves $dZ = \tilde{\sigma}(Z_t)dB_t$ for $\tilde{\sigma}: \mathbb{R} \to \mathbb{R}$ meas.
up to stopping the $\tau_M := \inf\{t \geq 0 : |Z_t| \geq M\}$, for all $M \geq 0$.
Then $\tau_M \xrightarrow[M\to\infty]{} \infty$ a.s.

In particular, Z has no blowups in finite time. (HW7)

So Z_t is defined for all $t \geq 0$ so $X_t = f^{-1}(Z_t)$ is as well.

For converse, assume Z is solution, and take $X_t := f^{-1}(Z_t)$.
check X solves its SDE by Itô. $\boxtimes$

This reduces time homogeneous SDE to "pure" SDE $dX_t = \sigma(X_t)dB_t$.

Issues with solving this:
- zeros of σ: If $\sigma(x_0) = 0$, $X_t := x_0$ is a solution?
  Q: Are there other sol's with $X_0 = x_0$?
- point x_0 where σ is small s.t. $\forall \varepsilon > 0$: $\int_{-\varepsilon}^{\varepsilon} \frac{1}{\sigma(x_0+y)^2} dy = +\infty$

Thm (Engelbert–Schmidt) Let $\sigma : \mathbb{R} \to \mathbb{R}$ be Borel.

Define $Z(\sigma) := \{x \in \mathbb{R} : \sigma(x) = 0\}$

$$I(\sigma) := \left\{ x \in \mathbb{R} : \forall \varepsilon > 0 : \int_{-\varepsilon}^{\varepsilon} \frac{1}{\sigma^2(x+y)^2} dy = +\infty \right\}$$

Then $dX_t = \sigma(X_t) dB_t$ admits a unique strong solution for every initial condition if and only if

$$I(\sigma) = Z(\sigma).$$

Special case solved in:

$\sigma : \mathbb{R} \to \mathbb{R}$ Borel s.t. $\frac{1}{\sigma^2} \in L^{1,loc}(\mathbb{R})$.

Thm Assume $\sigma : \mathbb{R} \to \mathbb{R}$. Then

Let W = SBM started at $W_0 = x_0$.

- $\forall t \geq 0 : U_t := \int_0^t \frac{1}{\sigma(W_s)^2} ds < \infty$ a.s.
- $U_t \xrightarrow[t \to \infty]{} +\infty$ a.s.

Set $T(u) := \inf\{t \geq 0 : U_t \geq u\}$, $B_u := \int_0^{T(u)} \frac{1}{\sigma(W_s)} dB_s$.

Then B is SBM and $X_u := W_{T(u)}$ solves $dX_t = \sigma(X_t) dB_t$ with $X_0 = x_0$.

For proof we need time change under Itô integral.

Let Z be adapted, jointly measurable w.r.t $\{\mathcal{F}_t\}$, $B = SBM$.

Thm Assume $U_t := \int_0^t Z_s^2 \, ds < \infty$ a.s. $\forall t < \infty$, $U_t \xrightarrow[t \to \infty]{} \infty$ a.s.

Denote $T(u) := \inf\{t \geq 0 : U_t \geq u\}$ (LC inverse)

$\tilde{B}_u := \int_0^{T(u)} Z_s \, dB_s$.

Then $\tilde{B}$ is SBM w.r.t. filtration $\{\mathcal{F}_{T(u)}\}_{u \geq 0}$.

Let $\{Y_t : t \geq 0\}$ LC process, adapted to $\{\mathcal{F}_t\}_{t \geq 0}$.

The $\{Y_{T(u)} ; u \geq 0\}$ is LC process, adapted to $\{\mathcal{F}_{T(u)}\}_{u \geq 0}$,

and $\forall u \geq 0$: $\int_0^u Y_{T(v)}^2 \, dv = \int_0^{T(u)} Y_s^2 Z_s^2 \, ds$

and, if these are finite a.s. $\forall u \geq 0$:

$$\forall u \geq 0 : \int_0^u Y_{T(u)} \, d\tilde{B}_u = \int_0^{T(u)} Y_s Z_s \, dB_s \quad \text{a.s.}$$

Shorthand: $du(t) = Z_t^2 \, dt$

$d\hat{B}_{u(t)} = Z_t \, dB_t$ ... $\int_0^{u(t)} Y_v \, d\hat{B}_v = \int_0^t Y_s Z_s \, dB_s$

Pf. • Let W = SBM, $\tau_M := \inf\{t \geq 0 : |W_t - x_0| \geq M\}$.

$$E\left(1_{\{\tau_M > t\}} \int_0^t \frac{1}{\sigma(W_s)^2} ds\right) \leq \int_0^t E\left(\frac{1}{\sigma(W_s)^2} 1_{|W_s - x_0| \leq M}\right) ds$$

$$\underset{W_s \text{ has PD} \leq \frac{1}{\sqrt{2\pi s}}}{=} \int_0^t \frac{ds}{\sqrt{2\pi s}} \int_{-M}^{M} \frac{1}{\sigma(x_0+y)^2} dy < \infty \text{ by assumption.}$$

Since $P(\bigcup_{M \geq 0} \{\tau_M > t\}) = 1$, we get $U_t < \infty$ a.s.

• $U_\infty := \lim_{t \to \infty} U_t$ exists. Setting $\tau := \inf\{t \geq 1 : W_t = x_0\}$

recurrence of SBM $\Rightarrow$ $\tau < \infty$ a.s. So U_∞ is the sum of infinite sequence of iid copies of $\int_0^{\tau_x} \frac{1}{\sigma(W_s)^2} ds$. $\Rightarrow U_\infty = +\infty$ a.s.

So, $\langle B \rangle_u = \int_0^{T(u)} \frac{1}{\sigma(W_s)^2} ds = U_{T(u)} = u$, B = SBM.

Time change under stoch integral:

$$\int_0^u \sigma(W_{T(u)}) dB_u = \int_0^{T(u)} \sigma(W_{T(v)}) \frac{1}{\sigma(W_{T(v)})} dW_v = W_{T(u)}$$

So $X_u := W_{T(u)}$ solves SDE. $\boxtimes$

Notes Combine time change + reduction to ODE to reduce $dX_t = a(t, X_t)dt + \sigma(t, X_t)dB_t$ to solution of two coupled ODEs.