# Tanaka eg & weak solutions & methods to solve SDEs

<u>Lemma</u> Let $B = $ SBM started at any $B_0$. Then

$$\forall x \in \mathbb{R}\ \forall t \geq 0: \quad \frac{1}{2\varepsilon}\int_0^t \mathbb{1}_{|B_s - x| < \varepsilon}\, ds \xrightarrow[\varepsilon \downarrow 0]{P} |B_t - x| - |B_0 - x| - \int_0^t \operatorname{sgn}(B_s - x)\, dB_s$$

when $\{\varepsilon_n\}_{n\geq 1}$ is s.t. $\sum_{n\geq 1} \varepsilon_n < \infty$, then limit a.s. along $\{\varepsilon_n\}_{n\geq 1}$.

$$\operatorname{sgn}(x) := \begin{cases} 1 & x \geq 0 \\ -1 & x < 0 \end{cases}$$

<u>Prop</u> (Tanaka) The SDE $dX_t = \operatorname{sgn}(X_t)\, dB_t$ has no strong solution with $X_0 = 0$
for filtration $\tilde{\mathcal{F}}_t^B = \sigma(N \cup \{B_s : s \leq t\})$, $N =$ null sets

<u>Intuition</u>: $X_t \geq 0 \Rightarrow \operatorname{sgn}(X_t) = +1 \ldots X$ follows $B$

After hit of zero, if "next" excursion $\geq 0$ .... keep following $B$
if —— " —— $< 0$ ... keep following $-B$

But 2nd part forces $X$ to be positive!

Problem: There is no "next" excursion!

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**Lemma** Let $B = $ SBM started at any $B_0$. Then

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**Intuition:** $X_t \geq 0 \Rightarrow \operatorname{sgn}(X_t) = +1 \ldots X$ follows $B$

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But $2^{nd}$ part forces $X$ to be positive!

Problem: There is no "next" excursion!

Pf: Suppose $X$ is strong solution with $X_0 = 0$, adapted to $\{\widetilde{\mathcal{F}}_t^B\}_{t\geq 0}$.

Then $\langle X \rangle_t = \int_0^t \operatorname{sgn}(X_s)^2 ds = \int_0^t ds = t \Rightarrow X = SBM$.

Moreover, $\int_0^t \operatorname{sgn}(X_s) dX_s = \int_0^t \operatorname{sgn}(X_s)^2 dB_s = \int_0^t dB_s = B_t$.

Lemma: $B_t = \int_0^t \operatorname{sgn}(X_s) dX_s \underset{\varepsilon_n = \frac{1}{n^2}}{\overset{\text{Lemma}}{=}} |X_t| - |X_0| - \lim_{n\to\infty} \frac{1}{2\varepsilon_n} \int_0^t 1_{|X_s| \leq \varepsilon_n} ds$ a.s.

Hence, $B_t$ is $\widetilde{\mathcal{F}}_t^{|X|}$-meas. So we get:

$$\left.\begin{array}{l} X \text{ strong sol} \Rightarrow \widetilde{\mathcal{F}}_t^X \subseteq \widetilde{\mathcal{F}}_t^B \\ \text{Lemma} + X = SBM \Rightarrow \widetilde{\mathcal{F}}_t^B \subseteq \widetilde{\mathcal{F}}_t^{|X|} \end{array}\right\} \forall t \geq 0: \widetilde{\mathcal{F}}_t^X \subseteq \widetilde{\mathcal{F}}_t^{|X|}$$

Lemma $\forall t > 0$: $\{X_t > 0\} \notin \widetilde{\mathcal{F}}_t^{|X|}$.

Pf $\mathcal{G}_t := \{ A \in \widetilde{\mathcal{F}}_t^X : E(1_A X_t) = 0\}$

Then $\mathcal{G}_t$ is $\sigma$-alg.

Moreover, $\{|X_s| \in C\} \in \mathcal{G}_t$ because $E\left(1_{\{|X_s| \in C\}} X_t\right) = E\left(1_{\{|X_s| \in C\}} (-X_t)\right)$ by Brownian symmetry.

since $N \subseteq \mathcal{G}_t \Rightarrow \widetilde{\mathcal{F}}_t^{|X|} \subseteq \mathcal{G}_t$.

Yet $E\left(1_{\{X_t > 0\}} X_t\right) \underset{t>0}{>} 0$ so $\{X_t > 0\} \notin \widetilde{\mathcal{F}}_t^{|X|}$. $\boxtimes$

Hence, $\widetilde{\mathcal{F}}_t^X \not\subseteq \widetilde{\mathcal{F}}_t^{|X|}$ so $X$ could not exist. $\boxtimes$

So we can't solve $dX_t = \mathrm{sgn}(X_t)\,dB_t$ with $B$ as given.

<u>Yet</u> we can solve it as follows:

$$\left.\begin{array}{l} \text{Let } X = \text{SBM started at } X_0 \\ \text{define } \tilde{B}_t = \int_0^t \mathrm{sgn}(X_s)\,dX_s \end{array}\right\} \text{ then } \begin{aligned} X_t &= X_0 + \int_0^t \mathrm{sgn}(X_s)^2\,dX_s \\ &= X_0 + \int_0^t \mathrm{sgn}(X_s)\,d\tilde{B}_s \end{aligned}$$

<u>Def</u> A <u>weak solution</u> to SDE ... is <u>quadruplet</u>

$$\Big( (\Omega, \mathcal{F}, P),\ \{\mathcal{F}_t\}_{t\geq 0},\ \{B_t : t\geq 0\},\ \{X_t : t \geq 0\} \Big)$$

s.t. $\{X_t : t\geq 0\}$ is a strong solution for setting $\Big((\Omega,\mathcal{F},P), \{\mathcal{F}_t\}_{t\geq 0}, \{B_s : s\geq 0\}, X_0\Big)$

Uniqueness of weak solution harder to check:

— have to check pathwise uniqueness for all standard settings?

<u>Prop</u> Let $X$ be a strong solution to $dX_t = \mathrm{sgn}(X_t)\,dB_t$ on <u>some</u> standard setting. Let $\tau_0 := \inf\{t \geq 0 : X_t = 0\}$. Then

$$\tilde{X}_t := \begin{cases} X_t & t \leq \tau_0 \\ -X_t & t \geq \tau_0 \end{cases}$$

is also a strong solution (for that standard setting).

**Pf** $\tilde X$ is cont. (because $X_{\tau_0}=0$ if $\tau_0<\infty$).

$\tilde X$ is adapted: because $\tau_0$ is stopping time.

$\tilde X$ is strong sol.: $\mathrm{sgn}(\tilde X_t) = \left[2\,1_{\tau_0>t} - 1\right]\mathrm{sgn}(X_t)$

so
$$\int_0^t \mathrm{sgn}(\tilde X_s)\,dB_s = \int_0^t \left[2\,1_{\tau_0>s} - 1\right]\mathrm{sgn}(X_s)\,dB_s$$
$$= 2\int_0^{t\wedge\tau_0} \mathrm{sgn}(X_s)\,dB_s - \int_0^t \mathrm{sgn}(X_s)\,dB_s$$
$$= \begin{cases} 2(\underset{0}{\underset{\|}{X_{\tau_0}}} - X_0) - (X_t - X_0) = -X_t - (2\overset{\tilde X_0}{\overset{\|}{X_0}} - X_0) = \tilde X_t - \tilde X_0 & (t > \tau_0) \\ X_t - X_0 = \tilde X_t - \tilde X_0 & \text{if } t < \tau_0 \end{cases}$$
$\boxtimes$

We can change sign in every excursion! $\rightsquigarrow$ uncountably many strong solutions!

Yet: they are all standard Brownian motions!..

<u>Def</u> We say that <u>uniqueness in law</u> holds for an SDE if for any two weak solution $X, \tilde X$:
$$\tilde X_0 \overset{\text{law}}{=} X_0 \Rightarrow \{X_t : t\ge 0\} \overset{\text{law}}{=} \{\tilde X_t : t \ge 0\}$$
on $(C[0,\infty), \mathcal{B}(C[0,\infty)))$.

# Abstract non-sense theory of Yamada & Watanabe (1971)

Key results:

$\boxed{\underline{\text{Thm}} \text{ Pathwise uniqueness} \Longrightarrow \text{uniqueness in law}}$

<u>Thm</u> Assume a weak solution exists & pathwise uniqueness holds.

<u>Then</u> $\exists \chi : \mathbb{R} \times C[0,\infty) \to C[0,\infty)$ s.t.

$$\forall t \geq 0 : \sigma(\chi_s : s \leq t) \subseteq \sigma(\{x_0 \in A_1\}, \{b_s \in A_2\} : A_1, A_2 \in \mathcal{B}(\mathbb{R}), s \leq t)$$

and for any standard setting $((\Omega, \mathcal{F}, P), \{\mathcal{F}_t\}_{t \geq 0}, \{B_t : t \geq 0\}, X_0)$

$$X_t = \chi(X_0, B)$$

is a strong solution.

<u>Pf</u> Notes

## Methods to solve SDEs

Consider SDE: $dX_t = a(t, X_t)\,dt + dB_t$

<u>Lemma</u> $X$ solves this SDE iff $V_t = X_t - B_t$ solves the ODE

$$\frac{dV_t}{dt} = a(t, V_t + B_t)$$

<u>Pf</u> $X_t - B_t = X_0 + \int_0^t a(s, X_s)\,ds = X_0 + \int_0^t a(s, V_s + B_s)\,ds$

E.g Bessel eq: $dX_t = \frac{d-1}{2X_t} dt + dB_t$

$\therefore$ solve $\frac{dV_t}{dt} = \frac{d-1}{2} \frac{1}{V_t + B_t}$ until $\tau_0 = \inf\{t \geq 0 : V_t + B_t = 0\}$

then set $X_t := V_t + B_t$.

E.g Carathéodory - Peano thm. gives solution of ODE $\frac{dV}{dt} = a(t, V_t + B_t) dt$

whenever $t \longmapsto a(t,x)$ is meas, $x \longmapsto a(t,x)$ continuous

and $\forall r > 0\ \exists\ m_r : \mathbb{R}_+ \to \mathbb{R}_+$ s.t. $\sup_{|x| \leq r} |a(t,x)| \leq m_r(t) \wedge m_r \in L^{1,loc}(\mathbb{R}_+)$

Under same condition, we can solve SDE $dX_t = a(t, X_t) + dB_t$