Uniqueness & locality ... Tanaka eg

Thm Assume standard setting for $dX_t = a(t,X_t)dt + \sigma(t,X_t)dB_t$ and (a,σ), $(\tilde{a},\tilde{\sigma})$ be two pairs satisfying Lipschitz etc condition of Itô's thm. Let $D \subseteq \mathbb{R}^d$ be non-empty, open and $t_0 \in (0,\infty]$ be s.t.

$$\forall t \in \mathbb{R} \cap [0,t_0]\ \forall x \in D: \ \tilde{a}(t,x) = a(t,x) \wedge \tilde{\sigma}(t,x) = \sigma(t,x)$$

Let $X, \tilde{X}$ be strong solution up to stopping times $T, \tilde{T}$, resp.

Let $\tau = t_0 \wedge T \wedge \tilde{T} \wedge \inf\{t \in [0, T\wedge\tilde{T}] : X_t \notin D \vee \tilde{X}_t \notin D\}$.

Then $\forall t \geq 0\ \exists C(t) \in (0,\infty)$

$$E\left(\sup_{s \leq t\wedge\tau} |X_s - \tilde{X}_s|^2\right) \leq C(t)\, E\left(|X_0 - \tilde{X}_0|^2\right)$$

In particular,

$$P(X_0 = \tilde{X}_0) = 1 \Rightarrow P(\forall t \leq \tau : X_t = \tilde{X}_t) = 1$$

This is a statement of uniqueness and locality

Pf Let $B_r(0) := \{x \in \mathbb{R}^d : |x| < R\}$, $\tau_r := \tau \wedge \inf\{t \in [0,\tau] : X_t \notin B_r(0) \vee \tilde{X}_t \notin B_r(0)\}$

Then $X_{\tau_r \wedge t} - \tilde{X}_{\tau_r \wedge t} = X_0 - \tilde{X}_0 + A_t + M_t$

where $A_t := \int_0^t 1_{\tau_r > s}\left(a(s,X_s) - a(s,\tilde{X}_s)\right) ds$

$M_t := \int_0^t 1_{\tau_r > s}\left(\sigma(s,X_s) - \sigma(s,\tilde{X}_s)\right) dB_s$.

$$E\left(\sup_{s\le t} A_s^2\right) \le t\int_0^t E\left(|a(s,X_s) - a(s,\tilde{X}_s)|^2\right) ds \overset{\text{Lipschitz}}{\le} tK^2 \int_0^t E\left(\sup_{u \le s\wedge\tau_r} |X_s - \tilde{X}_s|^2\right) ds$$

$$E\left(\sup_{s\le t} M_s^2\right) \overset{\text{Doob}}{\le} 4E(M_t^2) \overset{\text{Itô iso}}{\le} \int_0^t E\left(|\sigma(s,X_s) - \sigma(s,\tilde{X}_s)|^2\right) ds \le K^2 \text{ —— '' ——}$$

Denoting $g_r(t) := E\left(\sup_{s \le t\wedge\tau_r} |X_s - \tilde{X}_s|^2\right)$ we get

$$g_r(t) \le 3E\left(|X_0 - X_0|^2\right) + 3K^2(t+4)\int_0^t g_r(s)\, ds$$

Lemma (Gronwall) If $\gamma : \mathbb{R}_+ \to \mathbb{R}_+$ is $L^{1,loc}(\mathbb{R}_+)$ and $t_0 > 0$, $\alpha, \beta \ge 0$ are such that $\forall t \le t_0 : \gamma(t) \le \alpha + \beta\int_0^t \gamma(s)\, ds$. Then

$$\forall t \le t_0 : \gamma(t) \le \alpha e^{\beta t}$$

Pf of Grönwall Pick $\tilde{\alpha} > \alpha$, set

$$t_1 := t_0 \wedge \inf\{t \in [0,t_0] : \gamma(t) > \tilde{\alpha} e^{\beta t}\}$$

Because RHS cont $\Rightarrow t_1 > 0$. If $t_1 < t_0$, then

$$\gamma(t_1) \le \alpha + \beta \int_0^{t_1} \tilde{\alpha} e^{\beta s}\, ds = \alpha + \tilde{\alpha}(e^{\beta t_1} - 1) = \tilde{\alpha} e^{\beta t_1} - (\tilde{\alpha} - \alpha)$$

For $t > t_1$ we thus get $\gamma(t) \le \tilde{\alpha} e^{\beta t_1} + \underbrace{(\alpha - \tilde{\alpha}) + \int_{t_1}^{t} \gamma(s)\, ds}_{< 0 \text{ when } t - t_1 \text{ small}}$

Hence, $t_1 = t_0$ as required. $\boxtimes$

Hence we get

$$E\Big(\sup_{s \le \tau_n \wedge t} |X_s - \tilde{X}_s|^2\Big) = \lim_{r \to \infty} g_r(t)$$

$$\overset{\text{Grönwall}}{\le} 3E(|X_0 - \tilde{X}_0|^2)\, e^{3K^2(t+4)t}$$

So $C(t) := 3e^{3K^2(t+4)t}$

If $P(X_0 = \tilde{X}_0) = 1$ we get $P(\forall t \le \tau : X_t = \tilde{X}_t)$ $\boxtimes$

Def An SDE has pathwise uniqueness if for any two strong solutions $\{X_t : t \geq 0\}$, $\{\tilde{X}_t : t \geq 0\}$:

$$P(X_0 = \tilde{X}_0) = 1 \Rightarrow P(\forall t \geq 0 : X_t = \tilde{X}_t) = 1$$

Def Given D, pathwise uniqueness up to first exit from D holds for an SDE if any two strong solutions $\{X_{t \wedge \tau_{D^c}} : t \geq 0\}$ and $\{\tilde{X}_{t \wedge \tilde{\tau}_{D^c}} : t \geq 0\}$, where $\tau_{D^c} := \inf\{t \geq 0 : X_t \notin D\}$ and $\tilde{\tau}_D$ similarly, we have $P(X_0 = X_0) = 1 \Rightarrow P(\forall t \leq \tau_D \wedge \tilde{\tau}_D : X_t = \tilde{X}_t) = 1$

Corollary For all $d \in \mathbb{R}$, Bessel SDE $dX_t = \frac{d-1}{2X_t} dt + dB_t$ has a strong solution up to first hit of zero and pathwise uniqueness holds up to first hit of zero.

Pf $D_n := \{x \in \mathbb{R} : |x| > 1/n\}$. Then $x \mapsto \frac{d-1}{2x}$ is globally Lipschitz on D_n
$\Rightarrow$ strong solution up to $\tau_{D_n^c}$ exists and is pathwise unique
Now take $n \to \infty$ and observe $\tau_0 = \inf\{t \geq 0 : X_t = 0\} = \lim_{n \to \infty} \tau_{D_n^c}$. $\boxtimes$

Tanaka eg: $\mathrm{sgn}(x) := \begin{cases} 1 & x \geq 0 \\ -1 & x < 0 \end{cases}$ (my choice)

Prop The Tanaka SDE $dX_t = \mathrm{sgn}(X_t)\, dB_t$ admits no strong solution with $X_0 = 0$.

Lemma (Tanaka formula) Let $B = \text{SBM}$. Then

$\forall t \geq 0$: $$\frac{1}{2\varepsilon}\int_0^t \mathbb{1}_{\{|B_s| \leq \varepsilon\}}\, ds \xrightarrow[\varepsilon \downarrow 0]{P} |B_t| - |B_0| - \int_0^t \mathrm{sgn}(B_s)\, dB_s$$

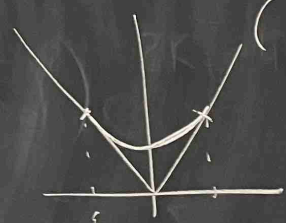
If $\{\varepsilon_n\}_{n \geq 1}$ is s.t. $\sum_{n \geq 1} \varepsilon_n < \infty$, then limit a.s. along this sequence.

Pf **Fact** Itô formula only needs a function $h \in C^1(\mathbb{R})$ s.t. $h' \in AC(\mathbb{R})$.

Then $h(B_t) = h(B_0) + \int_0^t h'(B_s)\, dB_s + \frac{1}{2}\int h''(B_s)\, ds$ (proof HW7)

We apply this to $h = f_\varepsilon$ where

$$f_\varepsilon(x) = \begin{cases} \frac{\varepsilon}{2} + \frac{x^2}{2\varepsilon} & |x| \leq \varepsilon \\ |x| & |x| \geq \varepsilon \end{cases}$$



Then $f_\varepsilon'(x) = \begin{cases} \frac{x}{\varepsilon} & |x| < \varepsilon \\ \mathrm{sgn}(x) & |x| \geq \varepsilon \end{cases}$, $f_\varepsilon''(x) = \frac{1}{\varepsilon}\mathbb{1}_{(-\varepsilon,\varepsilon)}(x)$

$$f_\varepsilon(B_t) = f_\varepsilon(B_0) + \int_0^t f_\varepsilon'(B_s)\,dB_s + \frac{1}{2\varepsilon}\int_0^t 1_{\{|B_s|<\varepsilon\}}\,ds$$

NTS: $\displaystyle\int_0^t f_\varepsilon'(B_s)\,dB_s \xrightarrow[\varepsilon \downarrow 0]{P} \int_0^t \mathrm{sgn}(B_s)\,dB_s$.

Note: $|f_\varepsilon'(x) - \mathrm{sgn}(x)| = 1_{(-\varepsilon,\varepsilon)}(x)\left|\frac{x}{\varepsilon} - \mathrm{sgn}(x)\right| \le 1_{(-\varepsilon,\varepsilon)}(x)$

$$E\left(\left(\int_0^t (f_\varepsilon'(B_s) - \mathrm{sgn}(B_s))\,dB_s\right)^2\right) \underset{\substack{\text{Itô iso}\\ \text{+ bound}}}{\le} E\left(\int_0^t 1_{|B_s|<\varepsilon}\,ds\right) = \int_0^t P(|B_s|<\varepsilon)\,ds$$

$$\le 2\varepsilon \int_0^t \frac{1}{\sqrt{2\pi s}}\,ds = 2\sqrt{\frac{2}{\pi}t}\;\varepsilon$$

by Chebyshev $\boxtimes$

$$\mathcal{L}_t(A) := \int_0^t 1_A(B_s)\,ds$$ occupation time measure

Tanaka: $\displaystyle\frac{\mathcal{L}_t([-\varepsilon,\varepsilon])}{2\varepsilon} \xrightarrow{P} |B_t| - |B_0| - \int_0^t \mathrm{sgn}(B_s)\,dB_s =: L_t(0)$

.... $\displaystyle\mathcal{L}_t(A) = \int_A L_t(x)\,dx$

Brownian local time