SDEs

$\mathcal{F}_0 \supseteq \{P \text{ null sets}\}$

Thm (Itô, 1940) Assume standard setting for $dX_t = a(t, X_t)\,dt + \sigma(t, X_t)\cdot dB_t$.
and, in addition, $X_0 \in L^2$ and $\exists K \in (0, \infty)$:

$$\forall t \geq 0 \ \forall x, y \in \mathbb{R}^d: \quad |a(t,x) - a(t,y)| + |\sigma(t,x) - \sigma(t,y)| \leq K|x-y|$$

$$\forall t \geq 0: \quad |a(t,0)| + |\sigma(t,0)| \leq K$$

Then a strong solution to above SDE exists.

Picard-Lindelöf:

$$X_t^0 := X_0 \ \wedge \ \forall n \geq 0: \ X_t^{(n+1)} := X_0 + \int_0^t a(s, X_s^{(n)})\,ds + \int_0^t \sigma(s, X_s^{(n)})\,dB_s.$$

Q: why do integrals exist?

Lemma: Under above conditions, $\forall n \geq 1 \ \forall t \geq 0$

$$\sup_{s \leq t} E\left((X_s^{(n)})^2\right) < \infty \ \wedge \ E\int_0^t |a(s, X_s^{(n)})|\,ds < \infty \ \wedge \ E\int_0^t |\sigma(s, X_s^{(n)})|^2\,ds < \infty$$

Pf.: Conditions imply $|a(t,x)| + |\sigma(t,x)| \le K(1+|x|)$

$\Rightarrow |a(t,x)|^2 + |\sigma(t,x)|^2 \le 2K^2(1+|x|^2)$

$(a+b)^2 \le 2a^2+2b^2$
$(a+b+c)^2 \le 3a^2+3b^2+3c^2$

$$E((X_s^{(n+1)})^2) \le 3E(X_0^2) + 3E\left(\left(\int_0^s |a(s,X_s^{(n)})|\,ds\right)^2\right) + 3E\left(\sup_{u\le s}\left|\int_0^u \sigma(s,X_s^{(n)})\,dB_s\right|^2\right)$$

$$\sup_{s\le t} E((X_s^{(n+1)})^2) \le 3E(X_0^2) + 3t\int_0^t E(|a(s,X_s^{(n)})|^2)\,ds \underset{\text{Itô iso}}{\overset{\text{Doob } L^2}{+}} 3\cdot 4\int_0^t E(|\sigma(s,X_s^{(n)})|^2)\,ds$$

$$\le 3E(X_0^2) + 3(t+4)2K^2\left[1 + \sup_{s\le t} E((X_s^{(n)})^2)\right]$$

so induction can proceed. $\boxtimes$

Lemma Set $g_n(t) := \begin{cases} E\left(\sup_{s\le t}|X_s^{(n)} - X_s^{(n-1)}|^2\right) & n\ge 1 \\ 4K^2(t+4)[1+E(X_0^2)] \end{cases}$

Then $g_n \in C(\mathbb{R}_+)$ and

$$\forall t\ge 0:\ g_{n+1}(t) \le 2K^2(t+4)\int_0^t g_n(s)\,ds$$

Pf Let $n \geq 1$. Then $X_t^{(n+1)} - X_t^{(n)} = A_t + M_t$

where $A_t := \int_0^t [a(s, X_s^{(n)}) - a(s, X_s^{(n-1)})]\, ds$

$M_t := \int_0^t [\sigma(s, X_s^{(n)}) - \sigma(s, X_s^{(n-1)})]\, dB_s$

(all stoch. integrals assume cont. version!)

So $\sup_{s \leq t} |X_s^{(n)} - X_s^{(n-1)}|^2 \leq 2 \sup_{s \leq t} A_s^2 + 2 \sup_{s \leq t} M_s^2$

Then $E\left(\sup_{s \leq t} A_s^2\right) \leq t\, E\left(\int_0^t |a(s, X_s^{(n)}) - a(s, X_s^{(n-1)})|^2 ds\right) \leq t K^2 \int_0^t E\left(\sup_{u \leq s} |X_s^{(n)} - X_s^{(n-1)}|^2\right) ds$

and $E\left(\sup_{s \leq t} M_s^2\right) \overset{\text{Doob}}{\leq} 4 E(M_t^2) \overset{\text{Itô}}{=} 4 E\left(\int_0^t |\sigma(s, X_s^{(n)}) - \sigma(s, X_s^{(n-1)})|^2 ds\right) \overset{\text{Lipschitz}}{\leq} 4K^2 \int_0^t E\left(\sup_{s \leq t} |X_s^{(n)} - X_s^{(n-1)}|^2 ds\right)$

So we get the claim for $n \geq 1$

For $n = 0$, the $(n-1)$-term is NOT there. Use the bound $|a(t,x)|^2 + |\sigma(t,x)|^2 \leq 2K^2(1+|x|^2)$

So we get

$g_1(t) \leq 2 \cdot 2K^2(t+4) E(1 + X_0^2)$ $\boxtimes$

Pf By induction: $\sup_{s\le t} g_n(t) \le 2 \frac{(2K^2(t+4))^{n+1}}{n!} s^n [1+E(X_0^2)]$

we kept integral to get this !

By Chebyshev: $P(\sup_{s\le t} |X_s^{(n)} - X_s^{(n-1)}| > 2^{-n}) \le 2 \cdot \frac{(2K^2(t+4))^{n+1}}{n!} (4t)^n E(1+X_0^2)$

Borel-Cantelli: $\Omega_0 := \{ \text{———} \; " \text{———} \text{ i.o. } \forall t \in \mathbb{N} \}^c \Rightarrow P(\Omega_0) = 1.$

Define:

$$X_t := \begin{cases} X_0 + \sum_{n=1}^{\infty} (X_t^{(n)} - X_t^{(n-1)}) & \text{on } \Omega_0 \\ X_0 & \text{on } \Omega \setminus \Omega_0 \end{cases} \qquad (\Omega_0 \in \mathcal{F}_0).$$

Locally uniform convergence $\Rightarrow$ $t \mapsto X_t$ is continuous.

Also: $\left[E\left(\sup_{s\le t} |X_s^{(n)} - X_s|^2\right)\right]^{1/2} \overset{\text{Fatou}}{\le} \lim_{m\to\infty} \sum_{j=n}^{m-1} \left[E\left(\sup |X_s^{(j+1)} - X_s^{(j)}|^2\right)\right]^{1/2}$

$\le \sum_{j\ge n} \sqrt{g_j(t)} \xrightarrow[n\to\infty]{} 0.$

Then Lipschitz estimates give

$$\int_0^t a(s, X_s^{(n)})\,ds \xrightarrow[n\to\infty]{a.s.} \int_0^t a(s, X_s)\,ds$$

$$\int_0^t \sigma(s, X_s^{(n)})\,dB_s \xrightarrow[n\to\infty]{P} \int_0^t \sigma(s, X_s) \cdot dB_s$$

So X is strong solution $\boxtimes$

Some issues:

- solution constructed simultaneously for a.e. Brownian path
- does solution see coefficients in parts of space it hasn't visited? (locality?)
- is solution unique?
- are uniform Lipschitz conditions and $X_0 \in L^2$ needed?

Def Assuming standard setting for SDE $dX_t = a(t,X_t)dt + \sigma(t,X_t)\,dB_t$ given a stopping time T, we say that a stoch process $\{X_{T\wedge t} : t \geq 0\}$ is strong solution up to time T if

(1) $\{X_{T\wedge t} : t\geq 0\}$ is adapted

(2) $\forall t \geq 0$: $\int_0^t 1_{\{T>s\}} |a(s,X_s)|\,ds < \infty \wedge \int_0^t 1_{\{T>s\}} |\sigma(s,X_s)|^2\,ds < \infty$ a.s.

(3) $\forall t \geq 0$: $X_{T\wedge t} = X_0 + \int_0^t 1_{\{T>s\}}\, a(s,X_s)\,ds + \int_0^t 1_{\{T>s\}}\, \sigma(s,X_s)\cdot dB_s$ a.s.

Thm (Uniqueness & locality) Assume standard setting with pairs (a,σ) and $(\tilde{a},\tilde{\sigma})$ satisfying Lipschitz/bdd conditions.

Let $D \subseteq \mathbb{R}^d$ be open and $t_0 > 0$ be s.t.

$$\forall t \le t_0 \ \forall x \in D: \quad \tilde{a}(t,x) = a(t,x) \ \wedge \ \tilde{\sigma}(t,x) = \sigma(t,x).$$

Given solution X up to stopping time T and $\tilde{X}$ up to stopping time $\tilde{T}$, let

$$\tau := t_0 \wedge T \wedge \tilde{T} \wedge \inf\{t \in [0, T \wedge \tilde{T}] : X_t \notin D \vee \tilde{X}_t \notin D\}.$$

Then $$\forall t \ge 0 \ \exists C(t) < \infty : \ E\Big(\sup_{s \le t \wedge \tau} |X_s - \tilde{X}_s|^2\Big) \le C(t)\, E\big(|X_0 - \tilde{X}_0|^2\big)$$

and in particular: $P(X_0 = \tilde{X}_0) = 1 \Rightarrow P(\forall t \le \tau : X_t = \tilde{X}_t) = 1.$

Note uniqueness fails without uniform Lipschitz, e.g. for Bessel eq.

existence for all times fails to

$$dX_t = (X_t^+)^2 dt + dB_t$$