Finishing Bessel processes

Thm Let $d \in \mathbb{R}$, $X = d$-dim. Bessel process, $\tau_a := \inf\{t \geq 0 : X_t = a\}$. Then $\forall x > 0$:

$\boxed{d \geq 2}$ $\quad \tau_0 = +\infty \ \wedge \ \inf_{t \geq 0} X_t > 0 \ \wedge \ \limsup_{t \to \infty} X_t = +\infty \quad P^x$-a.s.

$\boxed{d < 2}$ $\quad \tau_0 < +\infty \ \wedge \ \sup_{t \geq 0} X_t < \infty \quad P^x$-a.s.

$\boxed{d = 2}$ $\quad \tau_0 = +\infty \ \wedge \ \forall a > 0 : \tau_a < \infty \quad P^x$-a.s.

$\liminf_{t \to \infty} X_t = 0 \ \wedge \ \limsup_{t \to \infty} X_t = +\infty. \quad P^x$-a.s.

Pf. $\phi_d(x) = \begin{cases} x^{2-d} & d \neq 2 \\ \log(x) & d = 2 \end{cases}$ then $\forall \, \underline{0 < a < x < b} < \infty$: $P^x(\tau_a < \tau_b) = \dfrac{\phi_d(b) - \phi_d(x)}{\phi_d(b) - \phi_d(a)}$

continuity of X: $\quad \tau_b \xrightarrow[b \to \infty]{} \infty, \quad \tau_a \xrightarrow[a \downarrow 0]{} \tau_0$

$\boxed{d > 2}$ $\quad P^x(\tau_a < \infty) = \lim_{b \to \infty} P^x(\tau_a < \tau_b) = \left(\frac{x}{a}\right)^{2-d} \xrightarrow[a \downarrow 0]{} 0$

take $a_n \downarrow 0$ s.t. $P^x(\tau_{a_n} < \infty) \leq 2^{-n} \Rightarrow P^x(\tau_{a_n} < \infty \text{ i.o.}) = 0 \Rightarrow P^x(\exists n \geq 1 \, \forall t \geq 0 : X_t \geq a_n) = 1$

so $\inf_{t \geq 0} X_t > 0$ a.s., $\tau_0 = +\infty$ a.s.

$P^x(\tau_b < \infty) \overset{\tau_a \to +\infty}{=} \lim_{a \downarrow 0} P^x(\tau_b < \tau_a) = \lim_{a \downarrow 0} \dfrac{\phi_d(x) - \phi_d(a)}{\phi_d(b) - \phi_d(a)} = 1 \Rightarrow \limsup_{t \to \infty} X_t = +\infty.$

$\boxed{d<2}$ $P^x(\tau_0<\tau_b) = \lim_{a\downarrow 0} P^x(\tau_a<\tau_b) = 1-\left(\frac{x}{b}\right)^{2-d} \xrightarrow[b\to\infty]{} 1.$

So $P^x(\tau_0<\infty)=1$. Hence X is bdd a.s.

$\boxed{d=2}$ $P^x(\tau_a<\infty) = \lim_{b\to\infty} P^x(\tau_a<\tau_b) \overset{a>0}{=} \lim_{b\to\infty} \frac{\log b - \log x}{\log b - \log a} = 1$

So $P^x(\tau_0<\tau_b) = \lim_{a\downarrow 0} P^x(\tau_a<\tau_b) = \dots = 0$

Hence $P^x(\tau_0<\infty)=0$. $\boxtimes$

<u>Note</u> When $d>2$, we in fact have $\lim_{t\to\infty} X_t = +\infty$ P^x-a.s.

<u>Corollary</u> Let B = d-dimensional <u>SBM started at 0</u>, $d\in\mathbb{N}\setminus\{0\}$.

Then $\boxed{d=1}$ $B([0,\infty)) = \mathbb{R}$ (recurrence to points)

$\boxed{d=2}$ $\forall x\in\mathbb{R}^2\setminus\{0\}: P(\exists t\geq 0: B_t = x) = 0$

yet $B([0,\infty))$ is dense in $\mathbb{R}^2$ (F_σ-set)

(recurrence to balls)

$\boxed{d\geq 3}$ $\forall x\in\mathbb{R}^2\setminus\{0\}: \inf_{t\geq 0}|B_t - x|>0$ a.s.

(transience)

Stochastic differential equations (SDE)

An SDE is an expression of the form

$$dX_t = a(t,X_t)\,dt + \sigma(t,X_t)\,dB_t$$

Ex
- $dN_t = \alpha N_t\,dt + \beta N_t\,dB_t$ (population dynamics)
- $dX_t = \frac{d-1}{2X_t}\,dt + dB_t$ (Bessel process)
- $\underbrace{dX_t}_{\mathbb{R}^d\text{-valued}} = -\nabla V(X_t)\,dt + \underbrace{dB_t}_{\mathbb{R}^d\text{-valued}}$ (Langevin equation)

Note Last example is of the form

$$dX_t^{(i)} = a_i(t,X_t)\,dt + \sum_{j=1}^{m} \sigma_{ij}(t,X_t)\,dB_t^{(j)}$$

dot product

shorthand: $$dX_t = a(t,X_t)\,dt + \sigma(t,X_t)\cdot dB_t \quad (*)$$

m-dim SBM

d-dim. vector

d-dim. vector

d × m - matrix

Def A <u>standard setting for SDE ⊗</u> is quadruplet

$$\left((\Omega, \mathcal{F}, P), \{\mathcal{F}_t\}_{t\geq 0}, \{B_t : t\geq 0\}, X_0 \right)$$

where 1) $(\Omega, \mathcal{F}, P)$ prob. space with filtration $\{\mathcal{F}_t\}_{t\geq 0}$.

2) $\{B_t : t\geq 0\}$ = m-dim SBM adapted to $\{\mathcal{F}_t\}_{t\geq 0}$.

3) X_0 is $\mathbb{R}^d$-valued $\mathcal{F}_0$-meas. R.V.

and two Borel measurable functions:

$$a : \mathbb{R}_+ \times \mathbb{R}^d \to \mathbb{R}^d \quad \text{and} \quad \sigma : \mathbb{R}_+ \times \mathbb{R}^d \to \mathbb{R}^{dm}$$

Def A <u>strong solution to SDE</u> ⊗ is continuous stoch process $\{X_t : t\geq 0\}$ defined on $(\Omega, \mathcal{F}, P)$ s.t.

(1) X is adapted to $\{\mathcal{F}_t\}$

(2) $\forall t\geq 0$: $\int_0^t |a(s, X_s)| ds < \infty \wedge \int_0^t |\sigma(s, X_t)|^2 ds < \infty$ a.s.

(3) $\forall t \geq 0$: $X_t = X_0 + \int_0^t a(s, X_s) ds + \int_0^t \sigma(s, X_s) \cdot dB_s$ a.s.

Interpretation: $|a(t, X_t)| = \sum_{i=1}^d |a_i(t, X_t)|$, $|\sigma(t, X_t)|^2 = \mathrm{Tr}(\sigma \sigma^T) = \sum_{i,j} \sigma_{ij}(t, X_t)^2$

- $dX_t = -\nabla V(X_t)\,dt + dB_t$ (Langevin equation)

X_t: $\mathbb{R}^d$-valued; B_t: $\mathbb{R}^d$-valued

Ex population dynamics SDE

$$d\log(N_t) = \frac{1}{N_t}dN_t - \frac{1}{2N_t^2}d\langle N\rangle_t$$

$$= \frac{1}{N_t}(\alpha N_t\,dt + \beta N_t\,dB_t) - \frac{1}{2N_t^2}\beta^2 N_t^2\,dt$$

$$= (\alpha - \beta^2/2)\,dt + \beta\,dB_t$$

($N_t > 0$ so justified)

So

$$N_t = N_0 \exp\{(\alpha - \beta^2/2)t + \beta B_t\}$$

Ex (Bessel process) $X_t = |x + B_t|$ … $dX_t = \frac{d-1}{2X_t}dt + d\hat{B}_t$

B_t: d-dim SBM; $\hat{B}_t$: 1-dim

Not strong solution

Thm (Itô, 1940s) Assume standard setting for $\otimes$.
with $X_0 \in L^2$. and s.t. $\exists K \in (0,\infty)$:

$$\forall t \geq 0 \ \forall x, y \in \mathbb{R}^d: \ |a(t,x) - a(t,y)| + |\sigma(t,x) - \sigma(t,y)| \leq K|x-y|$$

and

$$\forall t \geq 0: \ |a(t,0)| + |\sigma(t,0)| \leq K.$$

Then a strong solution to $\otimes$ exists.

Pf based on Picard-Lindelöf iterations. For simplicity $m = 1 = d$

$$X_t^{(0)} := X_0 \qquad t \geq 0$$

$$X_t^{(n+1)} := X_0 + \int_0^t a(s, X_s^{(n)})\,ds + \int_0^t \sigma(s, X_s^{(n)})\,dB_s$$

Issue why is this well def?