Bessel processes

recall $B^{(1)}, \dots, B^{(d)}$ = iid SBM, $R_t := \left[\sum_{i=1}^{d} (B_t^{(i)})^2 \right]^{1/2}$

$$dR_t = \frac{d-1}{2R_t}\,dt + \sum_{i=1}^{d} \frac{1}{R_t} B_t^{(i)}\, dB_t^{(i)}$$

Lemma Set $\tilde{B}_t := \sum_{i=1}^{d} \int_0^t \frac{1}{R_s} B_s^{(i)}\, dB_s^{(i)}$.

Then $\tilde{B}$ is SBM.

Pf $\langle \tilde{B} \rangle_t \overset{\text{iid}}{=} \sum_{i=1}^{d} \int_0^t \frac{1}{R_s^2} (B_s^{(i)})^2\, ds = \int_0^t ds = t$. $\boxtimes$

Def Let $d \in \mathbb{R}$. A d-dimensional Bessel process is $[0,\infty)$-valued cont process $\{X_t : t \geq 0\}$ such that for a Brownian motion B and with $\tau_0 := \inf\{t \geq 0 : X_t = 0\}$ we have

$$\forall t \geq \tau_0 : X_t = 0 \;\wedge\; dX_t = \frac{d-1}{2X_t}\,dt + dB_t \text{ on } \{\tau_0 > t\}$$

Meaning: $X_t = X_0 + \int_0^t 1_{\tau_0 > s} \frac{d-1}{2X_s}\,ds + B_{t \wedge \tau_0} \qquad t \geq 0$

$\underline{Q}$: Existence (Yes for integer $d \geq 1$), else $\rightsquigarrow$ theory of SDE.
Uniqueness also next time ... not in general if $P^x(\tau_0 < \infty) > 0$.
yes by our choice of absorbing b.c.

$\underline{Q}$: When $\tau_0 < \infty$ or $\tau_0 = \infty$?

More generally, for diffusion

$$dX_t = v(X_t)\,dt + \sigma(X_t)\,dB_t$$

$\underline{Q}$: What $P^x(\tau_0 < \infty)$? where P^x = law of X with $P^x(X_0 = x)$.

$\underline{A}$: Harmonic analysis.

Let $\phi : \mathbb{R} \to \mathbb{R}$ be type $C^2(\mathbb{R})$. Then

$$d\phi(X_t) = \phi'(X_t)\,dX_t + \frac{1}{2}\phi''(X_t)\,d\langle X\rangle_t$$

$$= \left[\phi'(X_t)v(X_t) + \frac{1}{2}\phi''(X_t)\sigma(X_t)^2\right]dt + \phi'(X_t)\sigma(X_t)\,dB_t$$

Now assume that ϕ obeys ODE: $\phi'(x)v(x) + \frac{1}{2}\phi''(x)\sigma(x)^2 = 0$

Assuming $\sigma^2(\cdot) > 0$ we solve $\log \frac{\phi'(x)}{\phi'(x_0)} = -\int_{x_0}^{x} \frac{2v(z)}{\sigma^2(z)}\,dz$

$v, \sigma \in C(\mathbb{R})$

$$\underline{\phi(x_0)} \qquad \underline{\phi'(x_0)}$$

$$\phi(x) = \phi(x_0) + \phi'(x_0)\int_{x_0}^{x} \exp\left\{-2\int_{x_0}^{y} \frac{2v(z)}{\sigma(z)^2}\,dz\right\} dy$$

WANT $\phi'(x_0) \neq 0$

Lemma For X, ϕ as above and $a, b \in \mathbb{R}$ s.t. $a < b$:

Set $\tau_z := \inf\{t \geq 0 : X_t = z\}$.

Then $\forall x \in (a,b) : P^x(\tau_a \wedge \tau_b < \infty) = 1$

$$\forall x \in (a,b) : P^x(\tau_a < \tau_b) = \frac{\phi(b) - \phi(x)}{\phi(b) - \phi(a)}$$

Pf First part more difficult. ... time-change to SBM:

$$Z_t := \int_0^t 1_{\tau_a \wedge \tau_b > s} \phi'(X_s)\sigma(X_s)\,dB_s + (\tilde{B}_t - \tilde{B}_{\tau_a \wedge \tau_b}) 1_{\tau_a \wedge \tau_b \leq t}$$

where $\tilde{B}$ is SBM $\perp\!\!\!\perp X, B$.

$$\langle Z \rangle_t = \int_0^t \left[\underline{1_{\tau_a \wedge \tau_b > s}}\ \underline{\phi'(X_s)\,\sigma(X_s)^2} + \underline{1_{\tau_a \wedge \tau_b \leq s}} \right] ds$$

So $\exists c_1, c_2 \in (0, \infty) : \forall t \geq 0 : c_1 t \leq \langle Z \rangle_t \leq c_2 t$

Define $T(t) = \inf\{u \geq 0 : \langle Z_u \rangle \geq t\}$.

then $\frac{1}{c_2} t \leq T(t) \leq \frac{1}{c_1} t \quad (*) \quad \forall t \geq 0$.

Here $\{Z_{T(t)} : t \geq 0\}$ is SBM.

Observe: $Z_t = \phi(X_t)$ on $\{\tau_a \wedge \tau_b > t\}$.

So $\{\tau_a \wedge \tau_b = \infty\} = \{\forall t \geq 0 : Z_t \in (\phi(a) \wedge \phi(b), \phi(a) \vee \phi(b))\}$.

$\overset{(*)}{=} \{\forall t \geq 0 : Z_{T(t)} \in \text{——} \, \text{''} \, \text{——}\}$.

$P^x(\text{last event}) = 0$.

2nd part: ∘ $\{\phi(X_{t \wedge \tau_a \wedge \tau_b}) : t \geq 0\}$ is bounded martingale under P^x $\forall x \in (a,b)$.

∘ $\tau_a \wedge \tau_b < \infty$ P^x-a.s. $\forall x \in (a,b)$

OST: $M_t := \phi(X_{t \wedge \tau_b \wedge \tau_a})$.

$$\phi(x) \underset{x \in (a,b)}{\overset{?}{=}} E^x(M_0) \overset{OST}{=} E^x(M_{\tau_a \wedge \tau_b}) = P^x(\tau_a < \tau_b)\phi(a) + P^x(\tau_a > \tau_b)\phi(b)$$

$$= P^x(\tau_a < \tau_b)(\phi(a) - \phi(b)) + \phi(b)$$

$\boxtimes$

For Bessel equation: Let X = d-dimensional Bessel process: $dX_t = \frac{d-1}{2X_t} dt + dB_t$

$$\phi_d(x) := \begin{cases} x^{2-d} & d \neq 2 \\ \log(x) & d = 2 \end{cases}$$

$\boxed{\underline{\text{Lemma}} \ \forall\, 0<a<x<b<\infty: \{\phi_d(X_{t\wedge\tau_a\wedge\tau_b}) : t \geq 0\} \text{ is bdd martingale}}$

$\underline{\text{Pf}}$: $(d=2)$
$$d \log X_t = \frac{1}{X_t} dX_t - \frac{1}{2X_t^2} d\langle X\rangle_t$$
$$= \frac{1}{X_t^2}\left[\frac{d-1}{2} - \frac{1}{2}\right] dt + \frac{1}{X_t} dB_t$$

$d-1=1$
$\iff d=2$

$\underline{\text{Corollary}}$: For X = d-dim. Bessel process

$$\forall\, 0<a<x<b<\infty: \quad P^x(\tau_a<\tau_b) = \frac{\phi_d(b)-\phi_d(x)}{\phi_d(b)-\phi_d(a)}.$$

will now use to describe all cases separately:

Thm: Let $d \in \mathbb{R}$, $X = d$-dim. Bessel process. Then $\forall x > 0$:

$\boxed{d > 2}$: $\tau_0 = +\infty \wedge \inf_{t \geq 0} X_t > 0 \wedge \limsup_{t \to \infty} X_t = +\infty$ P^x-a.s.

$\boxed{d < 2}$: $\tau_0 < +\infty \wedge \sup_{t \geq 0} X_t < \infty$ P^x-a.s.

$\boxed{d = 2}$: $\tau_0 = +\infty \wedge \forall a > 0 : \tau_a < \infty$ P^x-a.s.

$\liminf_{t \to \infty} X_t = 0 \wedge \limsup_{t \to \infty} X_t = +\infty$ P^x-a.s.

Pf next time + notes