Finishing additive chaos theory

last time: $D_n := \{(t_1 \ldots t_n) \in \mathbb{R}^n : 0 \le t_1 < \cdots < t_n\}$

$$f \in L^{2,loc}(D_n) \longmapsto \{I_t^{(n)}(f) : t \ge 0\} \in \mathcal{M}_2^{cont}$$

Nesting $$I_t^{(n)}(f) = \int_0^t I_s^{(n-1)}(f_s)\, dB_s$$

where $f_s(t_1 \ldots t_{n-1}) = f(t_1, \ldots, t_{n-1}, s)$

Interpretation

$$I_t^{(n)}(f) = \int_0^t \left(\int_0^{t_{n-1}} \left(\cdots \left(\int_0^{t_2} f(t_1 \ldots t_n)\, dB_{t_1} \right) \cdots \right) dB_{t_{n-1}} \right) dB_{t_n}$$

Thm Fix $t \ge 0$, set $\mathcal{H}_0 := \{\text{constant r.v.'s}\}$
and for $n \ge 1$:

$$\mathcal{H}_n := \left\{ I_t^{(n)}(f) : f \in L^2(D_n \cap [0,t]^n) \right\}$$

Then $$L^2(\Omega, \mathcal{F}_t^B, P) = \bigoplus_{n \ge 0} \mathcal{H}_n \quad \ldots \quad (\forall n \ne m : \mathcal{H}_n \perp \mathcal{H}_m)$$

where $\{\mathcal{H}_n\}$ are orthogonal closed subspaces of $L^2(\Omega, \mathcal{F}_t^B, P)$

Lemma $\forall m \neq n\ \forall f \in L^{2,loc}(D_m)\ \forall g \in L^{2,loc}(D_n)\ \forall t,u \geq 0$:

$$E\left(I_t^{(m)}(f)\, I_s^{(n)}(g)\right) = 0$$

Pf $E\left(I_t^{(m)}(f) I_s^{(n)}(g)\right) \overset{\text{nesting + martingale}}{=} \int_0^{t \wedge s} E\left(I_u^{(m-1)}(f_u)\, I_u^{(n-1)}(g_u)\right) du$

Integrand vanishes for $m=1<n$ because $I_u^{(0)}(f_u) = \text{constant}(u)$

By induction, expectation vanishes whenever $1 \leq m < n$. $\boxtimes$

Pf of Thm: Construction of $I_t^{(n)}(f)$ + isometry $\Rightarrow$ $\mathcal{H}_n$ closed linear subspace of $L^2(\Omega, \mathcal{F}^B_t, P)$. Lemma $\Rightarrow$ $\forall m \neq n$: $\mathcal{H}_m \perp \mathcal{H}_n$.

NTS $X \perp \bigoplus_{n \geq 0} \mathcal{H}_n \Rightarrow X = 0$.

Will show: $\forall \ell \geq 1\ \forall t_1, \dots, t_\ell\ \forall n_1 \dots n_\ell \geq 1$: $B_{t_1}^{n_1} \cdots B_{t_\ell}^{n_\ell} \in \bigoplus_{k=0}^{n} \mathcal{H}_k$

Take f simple of form: $f(t_1 \dots t_n) = \prod_{k=1}^{n} 1_{(s_{j_{k-1}}, s_{j_k}]}(t_k)$ where $n = n_1 + \dots + n_\ell$. where $0 \leq s_1 < \dots < s_m = t$, $1 \leq j_1 < \dots < j_n \leq m$

Then $I_t^{(n)}(f) = \prod_{k=1}^{n} \left(B_{s_{j_k}} - B_{s_{j_{k-1}}}\right)$

and

$$B_t I_t^{(n)}(f) = \sum_{\ell=1}^{n} \left(B_{s_{j_\ell}} - B_{s_{j_{\ell-1}}}\right) \prod_{k=1}^{n} \left(B_{s_{j_k}} - B_{s_{j_{k-1}}}\right)$$

So $B_t I_t^{(n)}(f) = I_t^{(n)}(g)$ + terms involving $(B_{s_{j_0}} - B_{s_{j_0-1}})^2$

$\uparrow$ simple in $L^2(D_{n+1} \cap [0,t]^n)$

Notice: $(B_s - B_u)^2 \underset{s>u}{=} s - u + 2\int_s^u B_r \, dB_r$

$= s - u + 2\int_s^u \left(\int_s^r dB_v\right) dB_r$

Hence we get:

$\exists g \in L^2(D_{n+1} \cap [0,t]^{n+1})$ $\exists h \in L^2(D_{n-1} \cap [0,t]^{n-1})$ s.t.

$$B_t I_t^{(n)}(f) = I_t^{(n+1)}(g) + I_t^{(n-1)}(h)$$

Hence, proceeding inductively: $B_t I_t^{(n)}(f) \in \bigoplus_{k=0}^{n+1} \mathcal{H}_k$ when $f \in L^{2,loc}(D_n)$ is simple

Since $B_t \in L^q$ for every $q > 0$ we need $f_k \xrightarrow{L^2(D_n)} f \Rightarrow I_t(f_k) \xrightarrow{L^p} I_t(f)$ for some $p > 2$

Lemma (BDG-inequality) $\exists C_4 \in (0,\infty)$ s.t.

$\forall M \in \mathcal{M}_{loc}^{cont}$ $\forall t \geq 0$: $E(M_t^4) \leq C_4 E(\langle M \rangle^2)$

Hence, $E(I_t^{(n)}(f)^4) \leq C_4 E(\langle I^{(n)}(f) \rangle_t^2)$

$= C_4 E\left(\left[\int_0^t I_s^{(n-1)}(f_s)^2 ds\right]^2\right) \leq C_4^n \left(\int_0^t f(t_1 \ldots t_n)^2 dt_1 \ldots dt_n\right)^2$

From $B_t I_t^{(n)}(f) \in \bigoplus_{k=0}^{n+1} \mathcal{H}_k$ we get $B_{t_1}^{n_1} \cdots B_{t_\ell}^{n_\ell} \in \bigoplus_{k=0}^{n} \mathcal{H}_k$.

Usual argument $\Rightarrow$ $X \perp \bigoplus_{k=0}^{n} \mathcal{H}_k \Rightarrow E(X \mid \sigma(B_{t_1} \dots B_{t_n})) = 0$

$\forall t_1 \dots t_n \leq t$.

Lévy forward theorem: $E(X \mid \mathcal{F}_t^B) = 0$

So $X = 0$ proving $\bigoplus_{k \geq 0} \mathcal{H}_k = L^2(\Omega, \mathcal{F}_t^B, P)$. $\boxtimes$

Corollary: $\forall X \in L^2(\Omega, \mathcal{F}_t^B, P)$ $\exists \{h^{(k)}\}_{k \geq 1}$ with $h^{(k)} \in L^2(D_k \cap [0,t]^k)$

s.t. $X = EX + \sum_{k \geq 1} I_t^{(k)}(h^{(k)})$ (conv. in L^2)

Corollary $\forall X \in L^2(\Omega, \mathcal{F}_t^B, P)$ $\exists Y \in \mathcal{V}_B$:

$$X = EX + \int_0^t Y_s \, dB_s$$

Pf $Y_s := \sum_{k \geq 1} I_s^{(k-1)}(h_s^{(k)})$ conv in L^2 by isometry

(Need to pick convergent subsequence)

Then use $\int_0^t Y_s \, ds = \sum_{k \geq 1} \int_0^t I_s^{(k-1)}(h_s^{(k)}) \, dB_s = \sum_{k=1}^{\infty} I_t^{(n)}(h^{(k)}) = X - EX$

$\boxtimes$

Corollary: Let $h_n(x) := (-1)^n e^{x^2/2} \frac{d^n}{dx^n} e^{-x^2/2}$ … n-th Hermite polynomial

Then
$$h_n(B_t/\sqrt{t}) = \frac{n!}{t^{n/2}} \underbrace{\int_0^t \left(\int_0^{t_n} \left(\dots \left(\int_0^{t_2} 1 \, dB_{t_1} \right) \dots \right) dB_{t_n}\right)}_{I_t^{(n)}(1)}$$

$$n! \int_{t_1 < \dots < t_n} dB_{t_1} \dots dB_{t_n} \neq \left(\int_0^t dB_s \right)^n$$

subleading terms in h_n "correct" for non-commutation

$$e^{\int_0^t f(s)\, dB_s - \frac{1}{2}\int f(s)^2 ds} = 1 + \sum_{n \geq 0} I_t^{(n)}(\underbrace{f \otimes \cdots \otimes f}_{n})$$

$$e^{\lambda B_t - \frac{\lambda^2}{2} t} = 1 + \sum_{n \geq 0} h_n(B_t/\sqrt{t}) \frac{t^{n/2}}{n!} \lambda^n$$

$$= C_4 \, E\left(\left[\int_0^t I_s^{(n-1)}(f_s)^2 ds \right]^2 \right) \leq C_4^n \int_0^t f(t_1 \dots t_n)^4 dt_1 \cdots dt_n$$