Last time: $\forall X \in L^2(\Omega, \mathcal{F}, P)$ $\exists Y \in \mathcal{V}_B$ $\forall t \geq 0$: $E(X \mid \mathcal{F}_t^B) = EX + \int_0^t Y_s \, dB_s$ (Doob-Dynkin)

supports SBM B

<u>Corollary</u>: For M cont. L^2-martingale adapted to $\{\mathcal{F}_t^B\}_{t\geq 0}$ $\exists Y \in \mathcal{V}_B$ s.t.

$$\forall t \geq 0: \quad M_t = M_0 + \int_0^t Y_s \, dB_s$$

<u>Pf</u> For $n \geq 1$ natural: $E(M_n \mid \mathcal{F}_t^B) = M_{n\wedge t}$ a.s., also $EM_n = EM_0 = M_0$ a.s. because $P|_{\mathcal{F}_0^B}$ trivial.

This gives $M_{n\wedge t} = M_0 + \int_0^t Y_s^{(n)} \, dB_s$ for some $Y^{(n)} \in \mathcal{V}_B$.

Then $\int_0^n Y_s^{(n+1)} \, dB_s = \int_0^n Y_s^{(n)} \, dB_s$ a.s. $\overset{\text{Itô iso}}{\Longrightarrow}$ $\int_0^n (Y_s^{(n+1)} - Y_s^{(n)})^2 \, ds = 0$ a.s.

Set $Y_t := \sum_{n\geq 1} Y_s^{(n)} 1_{(n-1,n]}(s)$. Then $\forall t \geq 0$: $M_t = M_0 + \int_0^t Y_s \, dB_s$ $\boxtimes$

<u>Additive chaos theory</u>

Iterated Itô integral: Ex $\int_0^t \left(\int_0^s X_u \, dM_u \right) Y_s \, dN_s$ $\quad X \in \mathcal{V}_M^{loc}$, $Y \in \mathcal{V}_N^{loc}$

$\cancel{\int_0^t X_u \left(\int_u^t Y_s \, dN_s \right) dM_u}$

<u>Issues</u> • Fubini-Tonelli out? due to adaptability requirement

• You can't take $X_{s,u}$

Sol: Take deterministic integrand and consider iterated integrals of the form:

$$\int_0^t \left(\int_0^{t_n} \left(\int_0^{t_{n-1}} \cdots \left(\int_0^{t_2} f(t_1, \dots, t_n) \, dB_{t_1} \right) dB_{t_2} \cdots \right) dB_{t_{n-1}} \right) dB_{t_n}$$

Notation $D_n := \{(t_1 \dots t_n) \in \mathbb{R}^n : 0 \le t_1 < t_2 < \dots < t_n\}$.

$L^{2,loc}(D_n) := \{ f : \mathbb{R}^n \to \mathbb{R} \text{ Borel}, \ \forall t \ge 0 : \int_{D_n \cap [0,t]^n} f(t_1 \dots t_n)^2 dt_1 \dots dt_n < \infty \}$

For $f \in L^{2,loc}(D_n)$, $t \ge 0$ define $f_t \in L^2(\mathbb{R}^{n-1})$ by $f_t(t_1 \cdots t_{n-1}) := f(t_1, \dots, t_{n-1}, t)$

Assume B = standard B.M.

adapted to $\{\mathcal{F}_t\}_{t \ge 0}$ s.t. $\mathcal{F}_0 \supseteq \{P\text{-null sets}\}$.

Prop $\forall n \ge 1 \ \forall f \in L^{2,loc}(D_n) \ \exists \{ I_t^{(n)}(f) : t \ge 0 \} \in \mathcal{M}_2^{cont}$ with $I_0^{(n)}(f) = 0$ and s.t.

(1) if $n = 1$, then $\forall t \ge 0 : \ I_t^{(1)}(f) = \int_0^t f(s) dB_s$ a.s.

(2) if $n \ge 2$, then $\exists Y \in \mathcal{V}_B$ s.t.

$\forall t \ge 0 : \ Y_t = I_t^{(n-1)}(f_t) \ \wedge \ I_t^{(n)}(f) = \int_0^t Y_s \, dB_s$ a.s.

Moreover $f \mapsto I_t^{(n)}(f)$ is linear and obeys

$$E\left(I_t^{(n)}(f)^2\right) = \int_{D \cap [0,t]^n} f(t_1 \dots t_n)^2 dt_1 \dots dt_n$$

So it's an isometry of $L^2(D_n [0,t]^n) \to L^2$.

Pf <u>Idea</u> Define for f simple and take limits.

$s_1 < s_2 < \dots < s_m$

$$f(t_1 \dots t_n) := \sum_{0 \le j_1 < j_2 < \dots j_n \le m} a_{j_1 \dots j_n} \prod_{i=1}^{n} 1_{(s_{j_i-1}, s_{j_i}]}(t_i)$$

$$I_t^{(n)}(f) := \sum_{0 \le j_1 < \dots < j_n \le m} a_{j_1 \dots j_n} \prod_{i=1}^{n} \left(B_{s_{j_i} \wedge t} - B_{s_{j_i-1} \wedge t}\right)$$

$$E\left(I_t^{(n)}(f)^2\right) = \sum_{0 \le j_1 < \dots < j_n \le m} (a_{j_1 \dots j_n})^2 \prod_{i=1}^{n} \left(s_{j_i} \wedge t - s_{j_i-1} \wedge t\right) = \int_{D_n \cap [0,t]^n} f(t_1 \dots t_n)^2 \, dt_1 \dots dt_n$$

Above gives

$$I_t^{(n-1)}(f_t) = \sum_{0 \le j_1 \dots < j_n \le n} a_{j_1 \dots j_n} 1_{(s_{j_n-1}, s_{j_n}]}(t) \prod_{i=1}^{n-1} \left(B_{s_{j_i} \wedge t} - B_{s_{j_i-1} \wedge}\right)$$

Therefore $I_t^{(n)}(f) = \int_0^t I_s^{(n-1)}(f_s) \, dB_s$

Now take limits:

$\boxed{\text{Lemma The simple functions are dense in } L^{2,loc}(D_n)}$

Pf $h \in L^2(D_n \cap [0,t])$ suppose $h \perp \prod_{i=1}^{n} 1_{(s_{j_i-1}, s_{j_i}]}(t_i) \dots$ indicator of rectangle in $D_n \cap [0,t]^n$.
$\{A \in \mathcal{B}(D_n \cap [0,t]) : h \perp 1_A\}$ is σ-alg containing all rectangles so $= \mathcal{B}(D_n \cap [0,t]^n) \Rightarrow h = 0$. $\boxtimes$

Pick $f \in L^{2,loc}(D_n)$. Then $\exists f^{(k)}$ simple s.t. $\| f^{(k)} - f \|_{L^2(D_n \cap [0,t]^n)} \leq 16^{-k}$.

Then isometry: $E\left(\int_0^t \left| I_s^{(n-1)}(f_s^{(k+1)}) - I_s^{(n-1)}(f_s^{(k)}) \right|^2 ds \right) = \| f^{(k+1)} - f^{(k)} \|^2_{L^2(D_n \cap [0,t]^n)} \leq 4 \cdot 16^{-k}$

So $P\left(\mathrm{Leb}\left(s \leq t : \left| \underline{\qquad} \,\|\, \underline{\qquad} \right| > 2^{-k} \right) > 2^{-k} \right) \leq \frac{1}{8^{-k}} 4 \cdot 16^{-k} = 4 \cdot 2^{-k}$

Let $\Omega^* = \{$ this happens only finitely often $\}$, BC: $P(\Omega^*) = 1$.

Set $Y_s := \begin{cases} \limsup_{k \to \infty} I_s^{(n-1)}(f_s^{(k)}) \, 1 & \limsup \in \mathbb{R} \text{ on } \Omega^* \\ 0 & \text{else} \end{cases}$. Then $I_s^{(n)}(f_s^{(k)}) \longrightarrow Y_s$ a.e./a.s. L^2

L^2 conv.: $\int_0^t I_s^{(n-1)}(f_s^{(k)}) \, dB_s \xrightarrow{L^2} \int_0^t Y_s \, dB_s$

Ito. isometry $I_t^{(n)}(f^{(k)})$ converges and defines $I_t^{(n)}(f)$. Then $I_t^{(n)}(f) = \int_0^t Y_s \, dB_s$ holds a.s.

Next time: $\forall X \in L^2(\Omega, \mathcal{F}_t^B, P)$ $\exists \{ h^{(k)} \}_{k \geq 0}$ $h^{(k)} \in L^2(D_k \cap [0,t]^k)$ $\forall k \geq 0$.

s.t. $X = EX + \sum_{k \geq 1} I_t^{(k)}(h^{(k)})$