Kolmogorov Extension Thm & continuity

last time: $\{X_t : t \in T\}$ ... FDD's $\mu_{(t_1 \dots t_n)}(\cdot) = P((X_{t_1} \dots X_{t_n}) \in \cdot)$

Q: Do FDD's imply existence of X? ... consistency necessary!

Thm (Kolmogorov's Ext. Thm) Let $(\mathcal{X}, \Sigma)$ = standard Borel space, $\{\mu_{(t_1 \dots t_n)} : n \geq 1, t_1 \dots t_n \in T\}$ consistent family of measures, with above structure.
Then $\exists (\Omega, \mathcal{F}, P)$ = prob. space supporting $\{X_t : t \in T\}$ s.t.

$$\forall n \geq 1 \ \forall t_1 \dots t_n \in T \ \forall A \in \Sigma^{\otimes n} : \ P((X_{t_1} \dots X_{t_n}) \in A) = \mu_{(t_1 \dots t_n)}(A)$$

Thm (Hahn-Kolmogorov) Let $\mathcal{A}$ = algebra of subsets of Ω.
Let $\mu : \mathcal{A} \to [0, \infty]$ be a set function s.t. $\mu(\emptyset) = 0$ and

- μ is finitely additive $\forall \{A_i\}_{i=1}^n \in \mathcal{A}^n$ disjoint: $\mu(\bigcup_{i=1}^n A_i) = \sum_{i=1}^n \mu(A_i)$
- μ is countably subadditive $\forall \{A_k\}_{k \in \mathbb{N}} \in \mathcal{A}^{\mathbb{N}}$ disjoint: $\bigcup_{k \geq 1} A_k \in \mathcal{A} \Rightarrow \mu(\bigcup_{k \geq 1} A_k) \leq \sum_{k \geq 1} \mu(A_k)$

Then $\exists \bar{\mu} : \sigma(\mathcal{A}) \to [0, \infty]$ countably additive s.t. $\forall A \in \mathcal{A} : \bar{\mu}(A) = \mu(A)$.

We set: $S \subseteq T$

$$\begin{cases} \Omega := \mathcal{X}^T = \{\text{functions } T \to \mathcal{X}\} \\ \mathcal{F}_S := \sigma\left(\left\{ \bigtimes_{t\in T} A_t : \{A_t : t\in T\} \subseteq \Sigma,\ \{t\in T : A_t \neq \mathcal{X}\} \text{ is finite and } \subseteq S \right\}\right) \\ \mathcal{F} := \mathcal{F}_T = \Sigma^{\otimes T} \end{cases}$$

$\pi_S : \Omega \to \mathcal{X}^S$ is projection on coordinates in S

Lemma • $\mathcal{A} := \bigcup_{\substack{S \subseteq T \\ \text{finite}}} \mathcal{F}_S$ is an algebra

• $P(A) := \mu_{(t_1 \ldots t_n)}(\pi_S(A))$ $\quad A \in \mathcal{F}_S$ • finitely add. set function on $\mathcal{A}$

Note $\mathcal{F} = \sigma(\mathcal{A})$, all need to do is show P is count. subadditive

Lemma (Inner reg.) Let μ be a finite measure on standard Borel space $(\mathcal{X}, \Sigma)$. Then

$$\forall A \in \Sigma : \quad \mu(A) = \sup\{\mu(C) : C \subseteq A \text{ compact or } \emptyset\}$$

Lemma If $(\mathcal{X}, \Sigma)$ is standard Borel then $\forall n \geq 1$: $(\mathcal{X}^n, \Sigma^{\otimes n})$ is standard Borel

Lemma For $\mathcal{A}$ and P as above:

$$\forall \{B_n\}_{n\in\mathbb{N}} \in \mathcal{A}^{\mathbb{N}} : \quad B_n \downarrow \emptyset \implies P(B_n) \downarrow 0.$$

Pf by contrapositive Assume $B_n \downarrow \emptyset \ \wedge \ \varepsilon := \inf_{n\geq 1} P(B_n) > 0$

Assume WLOG: $\exists \underset{\text{distinct}}{\{t_n\}_{n\geq 1}}$ $\forall n \geq 1$: $B_n \in \mathcal{F}_{S_n}$ for $S_n := \{t_k : k=1..n\}$

By Lemmas:

$$\forall n\geq 1 \ \exists C_n' \subseteq \mathcal{X}^n \text{ compact}: \ \underset{C_n' \subseteq \pi_{S_n}(B_n)}{\wedge} \ \mu_{(t_1 \dots t_n)}\left(\pi_{S_n}(B_n) \setminus C_n'\right) < \varepsilon 2^{-n-1}$$

Lift C_n' to $C_n = \pi_{S_n}^{-1}(C_n')$ observe $C_n \subseteq B_n$.

Then
$$P\left(\bigcap_{k=1}^{n} C_k\right) \geq P\left(\bigcap_{k=1}^{n} B_k\right) - \sum_{k=1}^{n} P(B_k \setminus C_k)$$
$$\geq \varepsilon - \sum_{k=1}^{n} \varepsilon 2^{-k-1} \geq \frac{\varepsilon}{2} > 0.$$

Hence
$$\forall n \geq 1 : \bigcap_{k=1}^{n} C_k \neq \emptyset$$

By Cantor intersection property - based argument, (see Notes)
we construct $x \in \mathcal{X}^T$ s.t.

$$\forall n \geq 1: \quad x \in \bigcap_{k=1}^{n} C_k.$$

Hence, $\bigcap_{k \geq 1} C_k \neq \emptyset$ and so $\bigcap_{k \geq 1} B_k \neq \emptyset$. $\boxtimes$

<u>Pf of KET</u>: Let $\{A_k\}_{k \geq 1} \in \mathcal{A}^{\mathbb{N}}$ be disjoint with $\bigcup_{k \geq 1} A_k \in \mathcal{A}$.

Set $B_n := \bigcup_{k > n} A_k$. Then $\bigcup_{k \geq 1} A_k = \dot{\bigcup}_{k=1}^{n} A_k \;\dot{\cup}\; B_n$

Then $B_n \in \mathcal{A}$ and $B_n \downarrow \emptyset$. By Lemma: $P(B_n) \downarrow 0$.

Hence
$$P\Big(\bigcup_{k \geq 1} A_k\Big) \overset{\text{finite add.}}{=} \sum_{k=1}^{n} P(A_k) + P(B_n) \xrightarrow[n \to \infty]{} \sum_{k=1}^{\infty} P(A_k)$$

So P is cont. add. on $\mathcal{A}$.

By HK: P extends to a prob. measure on $(\Omega, \mathcal{F})$.

For X set $X_t(\omega) := \omega_t$ and check that X has right FDD's. $\boxtimes$

Q: Uniqueness of P extension?

A: • Suppose P, P' are two extensions.

Then $\mathcal{L} := \{ A \in \mathcal{F} : P(A) = P'(A) \}$

is (Dynkin's) <u>λ-system</u>

• $\mathcal{A}$ is closed under intersections ... <u>π-system</u>

<u>Thm</u> (π/λ-thm) Let $\mathcal{P} = \pi$-system, $\mathcal{L} = \lambda$-system. Then

$$\mathcal{P} \subseteq \mathcal{L} \Rightarrow \sigma(\mathcal{P}) \subseteq \mathcal{L}$$