Representation theorems.

showed: $\forall M \in \mathcal{M}^{\text{cont}}_{\text{loc}} \quad \exists B = \text{SBM}$ s.t. $M_t = B_{\langle M \rangle_t}$. $\{P\text{-null sets}\} \subseteq \mathcal{F}_0$.

Thm Let $M \in \mathcal{M}^{\text{cont}}_{\text{loc}}$ w.r.t. $\{\mathcal{F}_t\}_{t\geq 0}$ on $(\Omega, \mathcal{F}, P)$ s.t. $t \longmapsto \langle M \rangle_t$ AC a.s. Unless $\langle M \rangle$ strictly increasing, assume $(\Omega, \mathcal{F}, P)$ supports adapted SBM $\tilde{B}$ s.t. $\tilde{B} \perp\!\!\!\perp M$. Then there exists SBM B on $(\Omega, \mathcal{F}, P)$ adapted to $\{\mathcal{F}_t\}_{t\geq 0}$, and $\exists Y \in \mathcal{V}^{\text{loc}}_B$ s.t.

$$\forall t \geq 0: \quad M_t = M_0 + \int_0^t Y_s \, dB_s \quad \text{a.s.}$$

Moreover, Y is a version of $\left\{ \sqrt{\frac{d\langle M \rangle_s}{ds}} : s \geq 0 \right\}$.

Lemma (Sub rule for Itô integrals)

Let $M \in \mathcal{M}^{\text{cont}}_{\text{loc}}$, $X \in \mathcal{V}^{\text{loc}}_M$. Define $N \in \mathcal{M}^{\text{cont}}_{\text{loc}}$ by $N_t = \int_0^t X_s \, dM_s$.

Then for all $Y \in \mathcal{V}^{\text{loc}}_N$ we have

$$XY \in \mathcal{V}^{\text{loc}}_M \quad \wedge \quad \forall t \geq 0: \int_0^t Y_s \, dN_s = \int_0^t Y_s X_s \, dM_s \quad \text{a.s.}$$

In short: Subs rule $dN_t = X_t \, dM_t$ works for Itô integrals.

Pf $Y_s := Y_u \mathbb{1}_{(u,v]}(s)$, $Y_u \in L^\infty$, $\mathcal{F}_u$-meas., $X \in \mathcal{V}_0$.

Then $\int_0^t X_s Y_s \, dM_s = Y_u (N_{v \wedge t} - N_{u \wedge t}) = \int_0^t Y_s \, dN_s$

Now take $X^{(n)} \in \mathcal{V}_0$ s.t. $\int_0^t (X_s^{(n)} - X_s)^2 \, d\langle M \rangle_s \xrightarrow[n \to \infty]{P} 0$.

Then $\int_0^t X_s^{(n)} \, dM_s \xrightarrow{P} \int_0^t X_s \, dM_s = N_t$

Since $Y_u \in L^\infty$ also $\int_0^t Y_s^2 (X_s^{(n)} - X_s)^2 \, d\langle M \rangle_s \longrightarrow 0$ and so

$$\int_0^t Y_s X_s^{(n)} \, dM_s \longrightarrow \int_0^t Y_s X_s \, dM_s$$ so $\int_0^t X_s Y_s \, dM_s = \int_0^t Y_s \, dN_s$ as holds $\forall X \in \mathcal{V}_M^{loc}$.

Use linearity to extend to all $Y \in \mathcal{V}_0$, and play same game to extend to all $Y \in \mathcal{V}_N^{loc}$. $\boxtimes$

Pf of Thm: Let $\Omega_0 := \{ t \mapsto \langle M \rangle_t \text{ AC} \}$.

By cont of M, AC determined by cont-many conditions and so $\Omega_0 \in \mathcal{F}$. Assumption: $P(\Omega_0) = 1 \Rightarrow \Omega_0 \in \mathcal{F}_0$.

$\forall \varepsilon > 0 \; \exists \delta > 0 \; \forall I_1, \dots, I_n = (a_i, b_i]$ disjoint intervals $\sum_{i=1}^n |I_i| < \delta \Rightarrow \sum_{i=1}^n |M_{b_i} - M_{a_i}| < \varepsilon$

For $t > 0$ set $\tilde{Y}_t(\omega) := \liminf_{n \to \infty} n \left(\langle M \rangle_t(\omega) - \langle M \rangle_{t - 1/n}(\omega) \right)$

Set
$$Y_t(\omega) := \begin{cases} \tilde{Y}_t(\omega) & \text{if } \omega \in \Omega_0 \wedge t > 0 \wedge \tilde{Y}_t(\omega) < \infty \\ 0 & \text{else} \end{cases}$$

Then $Y \geq 0$, adapted s.t. $Y_t(\omega) = \frac{d\langle M\rangle_t(\omega)}{dt}$ wherever der. exists.

Lebesgue diff. thm: $\langle M\rangle_t(\omega) = \int_0^t Y_s(\omega)\, ds \quad \forall \omega \in \Omega_0$.

In particular, Y is a version of $\{ \frac{d\langle M\rangle_t}{dt} : t \geq 0 \}$.

Now let $$B_t := \int_0^t \frac{1}{\sqrt{Y_s}} \mathbb{1}_{Y_s > 0}\, dM_s + \int_0^t \mathbb{1}_{Y_s = 0}\, d\hat{B}_s$$

Then integrals well def (assuming cont. versions of integral) because

$$\langle B\rangle_t \overset{\tilde{B} \perp\!\!\!\perp M}{:=} \int_0^t \frac{1}{Y_s} \mathbb{1}_{Y_s>0}\, d\langle M\rangle_s + \int_0^t \mathbb{1}_{Y_s=0}\, ds$$

$$\overset{d\langle M\rangle_s = Y_s ds}{=} \int_0^t \frac{1}{Y_s} \mathbb{1}_{Y_s>0} Y_s\, ds + \int_0^t \mathbb{1}_{Y_s=0}\, ds = \int_0^t 1\, ds = t$$

So B is SBM!

Sub rule:

$$\int_0^t \sqrt{Y_s}\, dB_s = \int_0^t \sqrt{Y_s} \frac{1}{\sqrt{Y_s}} \mathbb{1}_{Y_s>0}\, dM_s + \int_0^t \sqrt{Y_s}\, \mathbb{1}_{Y_s=0}\, d\tilde{B}_s.$$

$$= \int_0^t \mathbb{1}_{Y_s>0}\, dM_s = M_t - M_0 - \int_0^t \mathbb{1}_{Y_s=0}\, dM_s$$

$$\left\langle \int_0 \mathbb{1}_{Y_s=0}\, dM_s \right\rangle_t = \int_0^t \mathbb{1}_{Y_s=0}\, d\langle M\rangle_s = \int_0^t \mathbb{1}_{Y_s=0} Y_s\, ds = 0.$$

Brownian martingales ... (cont. martingales adapted to Brownian filtration)

Thm Let $(\Omega, \mathcal{F}, P)$ support SBM $\{B_t : t \geq 0\}$. Set $\mathcal{F}_t^B := \sigma(B_s : s \leq t)$.
Let $X \in L^2(\Omega, \mathcal{F}, P)$. Then $\exists\, Y \in \mathcal{V}_B$ s.t.

$$\forall t \geq 0: \quad E(X \mid \mathcal{F}_t^B) = EX + \int_0^t Y_s \, dB_s \quad \text{a.s.}$$

Motivation Doob-Dynkin: X, Y are r.v., Y is $\sigma(X)$-meas $\Rightarrow Y = h(X)$

So $E(X \mid \mathcal{F}_t^B) = h(B_s : s \leq t)$.

Pf WLOG $EX = 0$. Project $E(X \mid \mathcal{F}_t^B)$ on $\left\{ \int_0^t Y_s \, dB_s : Y \in \mathcal{V}_B \right\}$.

For $Y \in \mathcal{V}_B$ define $\varphi(Y) := E\left(\left[E(X \mid \mathcal{F}_t) - \int_0^t Y_s \, dB_s \right]^2 \right)$.

Set $c := \inf\{\varphi(Y) : Y \in \mathcal{V}_B\} \in [0, \infty)$. Take minimizing sequence $\{Y^{(n)}\}$ and note

$$E\left(\left| \int_0^t Y_s^{(n)} \, dB_s - \int_0^t Y_s^{(m)} \, dB_s \right|^2 \right)$$

$$\leq 2\varphi(Y^{(n)}) + 2\varphi(Y^{(m)}) - 4\varphi\left(\frac{Y^{(n)} + Y^{(m)}}{2} \right) \xrightarrow[n,m \to \infty]{} 0$$

So by Itô isometry $\|Y^{(n)} - Y^{(m)}\|_{L^2([0,t]\times\Omega)} \xrightarrow[n\to\infty]{} 0$

Let Y be the limit process. Then $Y \longmapsto Y + \varepsilon Z$

$$\forall Z \in \mathcal{V}_B : \quad E\left(\underbrace{\left[E(X|\mathcal{F}_t^B) - \int_0^t Y_s \, dB_s \right]}_{=: V_t} \int_0^t Z_s \, dB_s \right) = 0$$

GOAL: Show $V_t = 0$ a.s.

For $0 \le t_0 < \dots \le t_n \le t$ $\quad \lambda_1 \dots \lambda_n \in \mathbb{R}$

$$M_s := \exp\left\{ i \underbrace{\sum_{j=1}^n \lambda_j (B_{t_j \wedge s} - B_{t_{j-1}\wedge s})}_{\int_0^t \sum_{j=1}^n \lambda_j 1_{(t_{j-1},t_j]}(s)\, dB_s} + \frac{1}{2} \sum_{j=1}^n \lambda_j^2 (t_j^{\wedge s} - t_{j-1}^{\wedge s}) \right\}$$

Itô formula: $dM_s = \underbrace{i \left(\sum_{j=1}^n \lambda_j 1_{(t_{j-1},t_j]}(s) \right)}_{Z_s} M_s \, dB_s$

Then $M_t = M_0 + \int_0^t Z_s \, dB_s$

and so $E(M_t V_t) = E(M_0 V_t) \overset{M_0 = 1}{=} E(V_t) = E(X) \overset{\text{assumed}}{=} 0$

Relabelling $\lambda_j' = \lambda_j - \lambda_{j+1}$ $\quad j = 1, \dots, n-1$, $\lambda_n' := \lambda_n$

we get $E\left(V_t e^{i \sum_{j=1}^n \lambda_j' B_{t_j}}\right) = 0 \Rightarrow \forall f \in L^2(\mathbb{R}^n) : E(V_t f(B_{t_1} \dots B_{t_n})) = 0$

$$\Rightarrow E(V_t | \sigma(B_{t_1} \dots B_{t_n})) = 0$$

$\Rightarrow \; E(V_t \mid \underbrace{\sigma(B_s : s \in \mathbb{Q} \cap [0,t])}_{= \mathcal{F}_t^B \text{ by continuity}}) = 0$ (by Levy Forward Thm)

$\Rightarrow \; V_t = 0$

which means $E(X \mid \mathcal{F}_t^B) = EX + \int_0^t Y_s \, dB_s$.

<u>Issues</u> Y depends on t !

Denote process constructed for $t = n$ by $Y^{(n)}$

Then $\forall t \le n$: $\int_0^t Y_s^{(n+1)} dB_s \overset{a.s.}{=} E\left(\int_0^{n+1} Y_s^{(n+1)} dB_s \,\middle|\, \mathcal{F}_t^B\right)$

$$= E\left(E(X \mid \mathcal{F}_{n+1}^B) - EX \,\middle|\, \mathcal{F}_t^B \right)$$

$$\overset{t \le n}{=} E\left(E(X \mid \mathcal{F}_n^B) - EX \,\middle|\, \mathcal{F}_t^B \right) \overset{a.s.}{=} \int_0^t Y_s^{(n)} dB_s$$

Hence, setting

$Y_t := \sum_{n \ge 1} Y_t^{(n)} \mathbb{1}_{(n-1,n]}(t)$ we get $\forall t \ge 0$: $E(X \mid \mathcal{F}_t^B) = EX + \int_0^t Y_s \, dB_s$ a.s.

$\boxtimes$