Representation theorems

Thm (Lévy's characterization of SBM)

Let $M \in \mathcal{M}^{cont}_{loc}$ be s.t. $M_0 = 0 \wedge \forall t \geq 0 : \langle M \rangle_t = t$.
Then M is standard Brownian motion.

Pf For $\lambda \in \mathbb{R}$ define $Z_t := e^{i\lambda M_t + \frac{\lambda^2}{2}\langle M \rangle_t}$.

Itô formula:

$$d Z_t = i\lambda Z_t \, dM_t + \frac{\lambda^2}{2} Z_t \, d\langle M \rangle_t + \frac{1}{2}(i\lambda)^2 Z_t \, d\langle M \rangle_t$$
$$= i\lambda Z_t \, dM_t$$

Hence, $Z_t = 1 + i\lambda \int_0^t Z_s \, dM_s$ a.s.

So $Z \in \mathcal{M}^{cont}_{loc}$.

For $\langle M \rangle_t = t$, $|Z_t| \leq e^{\frac{\lambda^2}{2}t} \Rightarrow Z \in \mathcal{M}^{cont}$.

$\left(\nwarrow \quad E(Z_{T_n \wedge t} \mid \mathcal{F}_s) = Z_{T_n \wedge s} \quad \{Z_{T_n \wedge s} : n \geq 1, s \leq t\} \text{ is UI} \right)$

So $\forall s \leq t :$ $E(Z_t \mid \mathcal{F}_s) = Z_s$ a.s.

$$\hookrightarrow E\left(e^{i\lambda M_t + \frac{\lambda^2}{2}t} \,\middle|\, \mathcal{F}_s\right) = e^{i\lambda M_s + \frac{\lambda^2}{2}s} \text{ a.s.}$$

Hence: $\forall s \leq t :$ $E\left(e^{i\lambda(M_t - M_s)} \,\middle|\, \mathcal{F}_s\right) = e^{-\frac{\lambda^2}{2}(t-s)}$ a.s.

Now take $0 \le t_0 < t_1 < \dots < t_n$, $\lambda_1 \dots \lambda_n \in \mathbb{R}$. Then

$$E\left(e^{i\sum_{j=1}^{n}\lambda_j (M_{t_j} - M_{t_{j-1}})}\right) = E\left(e^{i\sum_{j=1}^{n-1}\lambda_j (M_{t_j} - M_{t_{j-1}})}\, \underbrace{E\left(e^{i\lambda_n (M_{t_n} - M_{t_{n-1}})} \Big| \mathcal{F}_{t_{n-1}}\right)}_{e^{-\frac{\lambda_n^2}{2}(t_n - t_{n-1})}}\right)$$

$$= e^{-\frac{\lambda_n^2}{2}(t_n - t_{n-1})} E(\quad)$$

$$\overset{\text{induction}}{=} \exp\left\{-\sum_{j=1}^{n} \frac{\lambda_j^2}{2}(t_j - t_{j-1})\right\}$$

Cramér-Wold: $\{M_{t_j} - M_{t_{j-1}} : j = 1 \dots n\}$ are indep, $M_t - M_s = N(0, t-s)$.

Since $M_0 = 0 \wedge M$ cont $\Rightarrow$ M is SBM. $\boxtimes$

<u>Thm</u> Let $M \in \mathcal{M}_{loc}^{cont}$ be s.t. $M_0 = 0 \wedge t \longmapsto \langle M \rangle_t$ is strictly increasing with $\lim_{t \to \infty} \langle M \rangle_t = +\infty$. For $t \ge 0$ set

$$T(t) := \inf\{u \ge 0 : \langle M \rangle_u \ge t\}.$$

Then $\forall t \ge 0$: $T(t)$ = stopping time and $t \longmapsto T(t)$ is cont, strictly increasing with $T(t) \xrightarrow[t \to \infty]{} +\infty$. Moreover, $B_t := M_{T(t)}$, then B is SBM

and $\forall t \ge 0$: $M_t = B_{\langle M \rangle_t}$.

Proof requires:

Thm (Optional Stopping/Sampling Theorem) Let M be a RC martingale and T, S = stopping times s.t. $S \le T$.

Then $M_T \in L^1 \wedge E(M_T | \mathcal{F}_{S^+}) = M_S$ $\quad \mathcal{F}_{S^+} = \bigcap_{u>0} \mathcal{F}_{S+u}$

provided one of (1-3) hold:

(1) T is bounded

(2) M is bounded $\wedge$ $T < \infty$.

(3) $T < \infty$ and any other condition that ensures $\{M_{T \wedge t} : t \ge 0\}$ is UI.

In all these cases we then have $E(M_T) = E(M_S) = E(M_0)$.

Pf: Key fact to use: $\forall s \le t$: $E(M_t | \mathcal{F}_{T \wedge s}) = M_{T \wedge s}$

and so $\{M_{T \wedge s} : s \ge 0\}$ is UI under (1-3).

and $\{M_{T \wedge t} : t \ge 0\}$ is RC martingale.

(Proof easy for T discrete valued, then take limits ... HW5)

We can henceforth assume

$\{M_{T \wedge t} : t \ge 0\}$ is UI RC-martingale

Let $A \in \mathcal{F}_{S^+}$. Discretize $T_n := 2^{-n}\lceil 2^n T \rceil$, $S_n := 2^{-n}\lceil 2^n S \rceil$

Then T_n, S_n stopping times, $T_n \downarrow T$, $S_n \downarrow S$, $S_n \le T_n$.

Then
$$E(M_{T_n \wedge t} 1_A) = \sum_{k \ge 0} E(M_{T_n \wedge t} 1_{\underbrace{A \cap \{S_n = k2^{-n}\}}_{A \cap \{S_n \le k2^{-n}\} \setminus (A \cap \{S_n \le (k-1)2^{-n}\}) \in \mathcal{F}_{k2^{-n}}}})$$
$$= \sum_{k \ge 0} E(E(M_{T_n \wedge t} | \mathcal{F}_{k2^{-n}}) 1_{A \cap \{S_n = k2^{-n}\}})$$
$$= \sum_{k \ge 0} E(M_{\underbrace{T_n \wedge k2^{-n}}_{= k2^{-n} = S_n} \wedge t} 1_{A \cap \{S_n = k2^{-n}\}}) = E(M_{S_n \wedge t} 1_A)$$

So taking $n \to \infty$, $t \to \infty$, we get $\forall A \in \mathcal{F}_{S^+}$: $E(M_T 1_A) = E(M_S 1_A)$

hence $E(M_T | \mathcal{F}_{S^+}) = M_S$ a.s. $\boxtimes$

<u>Pf of Thm:</u> Note $\{T(t) \le u\} = \{\langle M \rangle_u \le t\} \in \mathcal{F}_u \Rightarrow T(t)$ = stopping time

we have $\langle M \rangle_{T(t)} = t$ $\forall t \ge 0$.

Now $Z_{T(t) \wedge u} = e^{i\lambda M_{T(t) \wedge u} + \frac{\lambda^2}{2} \langle M \rangle_{T(t) \wedge u}}$ is still cont. loc. martingale

with $|Z_{T(t) \wedge u}| \le e^{\frac{\lambda^2}{2} \langle M \rangle_{T(t) \wedge u}} \le e^{\frac{\lambda^2}{2} \langle M \rangle_{T(t)}} = e^{\frac{\lambda^2}{2} t}$

OST: $\forall s \le t$: $E(Z_{T(t)} | \mathcal{F}_{T(s)}) = Z_{T(s)}$

which reads $E(e^{i\lambda(M_{T(t)} - M_{T(s)})} | \mathcal{F}_{T(s)}) = e^{-\frac{\lambda^2}{2}(t-s)}$ $\boxtimes$

So $\{B_t = M_{T(t)} : t \ge 0\}$ is SBM and $M_t = B_{\langle M \rangle_t}$.

Thm Let $M \in \mathcal{M}^{cont}_{loc}$ w.r.t. $\{\mathcal{F}_t\}_{t\geq 0}$ s.t. $\mathcal{F}_0 \supseteq \{P\text{-null sets}\}$

Unless $t \mapsto \langle M \rangle_t$ is strictly increasing a.s., assume then the prob. space contains an indep. SBM.

Assume also $t \longmapsto \langle M \rangle_t$ is AC with $\lim_{t\to\infty} \langle M \rangle_t = +\infty$ a.s.

Then there exists a SBM $\{B_t : t \geq 0\}$ and a version of $\left\{\frac{d\langle M \rangle_t}{dt} ; t \geq 0\right\}$ s.t. $\left\{\sqrt{\frac{d\langle M \rangle_t}{dt}} ; t \geq 0\right\} \in \mathcal{V}_B$

s.t. $\forall t \geq 0$: $\quad M_t = M_0 + \int_0^t \sqrt{\frac{d\langle M \rangle_s}{ds}}\, dB_s$ a.s.

So: Cont. loc. martingales M with $\langle M \rangle$ AC are Itô integrals wrt SBM!