Localization & properties of Itô integral

last time: defined $M \in \mathcal{M}_2^{\text{cont}}$, $Y \in \mathcal{V}_0^{[\![\cdot]\!]_M} \longmapsto \int_0^t Y_s \, dM_s$

Want to extend to Y is

$$\boxed{\left\langle \int_0^{\cdot} Y_s \, dM_s \right\rangle_t = \int_0^t Y_s^2 \, d\langle M \rangle_s}$$

$$\mathcal{V}_M^{\text{loc}} := \left\{ Y : \begin{array}{l} \text{adapted, measurable} \\ \text{progressively measurable} \end{array} \wedge \ \forall t \geq 0 : \int_0^t Y_s^2 \, d\langle M \rangle_s < \infty \text{ a.s.} \right\}$$

locally integrable processes

Then:

Thm Let $M \in \mathcal{M}_{\text{loc}}^{\text{cont}}$, $Y \in \mathcal{V}_M^{\text{loc}}$. Set

$$\tau_K := \inf \left\{ t \geq 0 : \langle M \rangle_t \geq K \vee \int_0^t Y_s^2 \, d\langle M \rangle_s \geq K \right\}$$

Set $M_t^{(K)} := M_{\tau_K \wedge t}$. Then $M^{(K)} \in \mathcal{M}_2^{\text{cont}} \wedge Y \in \mathcal{V}_0^{[\![\cdot]\!]_{M^{(K)}}}$.

and $\forall L \geq K \geq 1 \ \forall t \geq 0$: $\int_0^t Y_s \, dM_s^{(L)} = \int_0^t Y_s \, dM_s^{(K)}$ a.s. on $\{\tau_K > t\}$.

Hence, $\int_0^t Y_s \, dM_s := \lim_{K \to \infty} \int_0^t Y_s \, dM_s^{(K)}$ exists a.s.

Moreover, integral admits a cont. version $\{I_t : t \geq 0\} \in \mathcal{M}_{\text{loc}}^{\text{cont}}$ s.t. $\forall T$ = stopping time

$$\forall t \geq 0: \int_0^{T \wedge t} Y_s \, dM_s := I_{T \wedge t} = \int_0^t Y_s \mathbb{1}_{T > s} \, dM_s = \int_0^t Y_s \, dM_{T \wedge s}$$

Proof same as for Brownian motion.
Note We truncate M instead of Y.

Lemma Let $M \in \mathcal{M}^{cont}_{loc}$, $Y, Y^{(n)} \in \mathcal{V}^{loc}_M$. Then $\forall t \geq 0$:

$$\int_0^t (Y_s^{(n)} - Y_s)^2 \, d\langle M\rangle_s \xrightarrow[n\to\infty]{P} 0 \Rightarrow \int_0^t Y_s^{(n)} \, dM_s \xrightarrow[n\to\infty]{P} \int_0^t Y_s \, dM_s$$

Lemma Let $M, M^{(n)} \in \mathcal{M}^{cont}_{loc}$, $Y \in \mathcal{V}^{loc}_M \cap \bigcap_{n\geq 1} \mathcal{V}^{loc}_{M^{(n)}}$. Then $\forall t \geq 0$:

$$\int_0^t Y_s^2 \, d\langle M^{(n)} - M\rangle_s \xrightarrow[n\to\infty]{P} 0 \Rightarrow \int_0^t Y_s \, dM_s^{(n)} \xrightarrow[n\to\infty]{P} \int_0^t Y_s \, dM_s$$

Properties of Itô integrals

Lemma (Linearity in Y) $\forall M \in \mathcal{M}^{cont}_{loc}$, $\forall Y, \tilde{Y} \in \mathcal{V}^{loc}_M$ $\forall \alpha, \beta \in \mathbb{R}$:

$\alpha Y + \beta \tilde{Y} \in \mathcal{V}^{loc}_M \wedge \forall t \geq 0$: $\int_0^t (\alpha Y_s + \beta \tilde{Y}_s) \, dM_s = \alpha \int_0^t Y_s \, dM_s + \beta \int_0^t \tilde{Y}_s \, dM_s$ a.s.

Lemma (Linearity in M) $\forall M, \hat{M} \in \mathcal{V}^{cont}_{loc}$ $\forall Y \in \mathcal{V}^{loc}_M \cap \mathcal{V}^{loc}_{\hat{M}}$ $\forall \alpha, \beta \in \mathbb{R}$:

$\alpha M + \beta \hat{M} \in \mathcal{V}^{cont}_{loc} \wedge \forall t \geq 0$: $\int_0^t Y_s \, d(\alpha M + \beta \hat{M})_s = \alpha \int_0^t Y_s \, dM_s + \beta \int_0^t Y_s \, d\hat{M}_s$ a.s.

Def A continuous semimartingale is a stoch process of the form $X_t = A_t + M_t$ where $M \in \mathcal{M}_{loc}^{cont}$ and A is continuous, adapted process s.t. $A_0 = 0 \wedge \forall t \geq 0: V_t^{(1)}(A) < \infty$.

Ex Diffusions: $A_t = \int_0^t U_s ds$, $M_t = X_0 + \underbrace{\int_0^t Y_s dM_s}_{\text{(Brownian martingale)}}$.

Note Decomposition of X unique up to indistinguishability.

So for $Y \in \mathcal{V}_M^{loc} \cap \{\forall t \geq 0: \int_0^t |Y_s||dA_s| < \infty \text{ a.s.}\}$

we can set

$$\int_0^t Y_s dX_s := \int_0^t Y_s dA_s + \int_0^t Y_s dM_s$$

integral w.r.t. signed measure

with this expression unige a.s.

Denote $\mathcal{S}_{cont} := \{X : \text{cont. semimartigale}\}$.

$$\langle X \rangle_t := \langle M \rangle_t \text{ when } X = A + M$$

Recall $\|\Pi\| \to 0$

$V_t^{(2)}(M, \Pi) \xrightarrow{L^2} \langle M \rangle_t$

So $V_t^{(2)}(X, \Pi) \xrightarrow{L^2} \langle X \rangle_t$

Thm (Itô formula) $\forall X \in \mathcal{S}_{cont}$, $\forall f \in C^2(\mathbb{R})$ $\forall t \geq 0$:

$$f(X_t) = f(X_0) + \int_0^t f'(X_s)\,dX_s + \frac{1}{2}\int_0^t f''(X_s)\,d\langle X\rangle_s.$$

Pf Same as for diffusions.
Integrals exist by continuity of $t \mapsto f^{(i)}(X_t)$.

Multivariate version requires:

Def Let $X, Y \in \mathcal{S}_{cont}$. The <u>covariation</u> $\langle X, Y\rangle$ is defined by

$$\langle X, Y\rangle_t := \frac{1}{4}\langle X+Y\rangle_t - \frac{1}{4}\langle X-Y\rangle_t.$$

Lemma $\forall M, N \in \mathcal{M}_{loc}^{cont}$, $\langle M, N\rangle$ is the unique adapted continuous process with $\langle M, N\rangle_0 = 0 \wedge \forall t \geq 0: V_t^{(1)}(\langle M, N\rangle) < \infty$ s.t. $\{M_t N_t - \langle M, N\rangle_t : t \geq 0\} \in \mathcal{M}_{loc}^{cont}$

Pf $M_t N_t = \frac{1}{4}(M_t + N_t)^2 - \frac{1}{4}(M_t - N_t)^2$

Lemma (Product rule / IBP) $\forall X, \tilde{X} \in \mathcal{S}_{cont}$:

$$\{X_t \tilde{X}_t : t \geq 0\} \in \mathcal{S}_{cont} \wedge d(X_t \tilde{X}_t) = \tilde{X}_t dX_t + X_t d\tilde{X}_t + d\langle X, \tilde{X}\rangle_t$$

Pf BV parts ... usual formula

For martingale part: $M_t \tilde{M}_t = \frac{1}{4}(M_t + \tilde{M}_t)^2 - \frac{1}{4}(M_t - \tilde{M}_t)^2$

... reduces to $\tilde{M} = M$

$$d(M_t^2) \overset{\text{Itô}}{=} 2 M_t dM_t + \frac{1}{2} \cdot 2\, d\langle M\rangle_t \qquad \boxtimes$$

Integral version:

$$\int_0^t \hat{X}_t dX_t = \tilde{X}_t X_t - \tilde{X}_0 X_0 - \int_0^t X_t d\tilde{X}_t - \langle \tilde{X}, X\rangle_t$$

There is a way to "absorb" Itô correction:

Def (Fisk-Stratonovich integral) $X, \tilde{X} \in \mathcal{S}_{cont}$:

$$\int_0^t \tilde{X}_s \circ dX_s := \int_0^t \hat{X}_s dX_s + \frac{1}{2}\langle \hat{X}, X\rangle_t$$

Lemma: Let $X \in \mathcal{S}_{\text{cont}}$, $f \in C^1(\mathbb{R})$, $t \geq 0$. Then $f \circ X \in \mathcal{S}_{\text{cont}}$

$$\langle f \circ X, X \rangle_t = \int_0^t f'(X_s)\, d\langle X \rangle_s .$$

In particular, the chain rule / "FTC" holds

$$f(X_t) = f(X_0) + \int_0^t f'(X_s) \circ dX_s$$

and product rule / IBP holds:

$$\forall \tilde{X} \in \mathcal{S}_{\text{cont}}: \int_0^t \tilde{X}_s \circ dX_s = \tilde{X}_t X_t - \tilde{X}_0 X_0 - \int_0^t X_s \circ d\tilde{X}_s$$

Thm (Multivariate Itô formula) Let $X^{(1)}, \dots, X^{(d)} \in \mathcal{S}_{\text{cont}}$ and $f : \mathbb{R}_+ \times \mathbb{R}^d \to \mathbb{R}$ be of type $C^1 \times C^2$. Then for $X_t := (X_t^{(1)}, \dots, X_t^{(d)})$:

$$\forall t \geq 0: \quad f(t, X_t) = f(0, X_0) + \int_0^t \frac{\partial f}{\partial s}(s, X_s)\, ds + \sum_{i=1}^d \int_0^t \frac{\partial f}{\partial x_i}(s, X_s)\, dX_s + \frac{1}{2} \sum_{i,j=1}^d \int_0^t \frac{\partial^2 f}{\partial x_i \partial x_j}(s, X_s)\, d\langle X^{(i)}, X^{(j)} \rangle_s$$