Integrability criteria & localization

recall: $Y \in \overline{\mathcal{V}_0}^{[\![\cdot]\!]_M}$, $M \in \mathcal{M}_2^{cont} \longmapsto \int_0^t Y_s \, dM_s$

$= \{ Y \in \mathcal{V}_M : (\exists \{Y^{(n)}\}_{n\in\mathbb{N}} \in \mathcal{V}_0^{\mathbb{N}} : [\![Y - Y^{(n)}]\!]_M \to 0 \}$

$$\sqrt{E\left(\int_0^t (Y_s - \tilde{Y}_s)^2 \, d\langle M \rangle_s \right)} = \text{"distance of } Y \text{ to } \tilde{Y} \text{ on } [0,t]\text{"}$$

Q: For what $Y \in \mathcal{V}_M$ we have $Y \in \overline{\mathcal{V}_0}^{[\![\cdot]\!]_M}$?

A: Depends on whether $t \longmapsto \langle M \rangle_t$ is AC a.s. or not.

Prop $\forall M \in \mathcal{M}_2^{cont}$ $\forall Y \in \mathcal{V}_M$: $t \longmapsto \langle M \rangle_t$ AC a.s. $\Rightarrow Y \in \overline{\mathcal{V}_0}^{[\![\cdot]\!]_M}$

Pf Assume first Y is bdd, say $\forall t \geq 0 : |Y_t| \leq K$.

Previous Thm: $\exists \{Y^{(n)}\}_{n\in\mathbb{N}} \in \mathcal{V}_0^{\mathbb{N}}$ $\forall t \geq 0$: $\underset{(\forall n \geq 1)}{|Y_t^{(n)}| \leq K} \wedge E\left(\int_0^t (Y_s^{(n)} - Y_s)^2 \, ds \right) \xrightarrow[n\to\infty]{} 0$

Borel-Cantelli: $\text{Leb}\left(\{ t \geq 0 : \limsup_{n\to\infty} |Y_t^{(n)} - Y_t| > 0 \} \right) = 0$ a.s.

Now (RNT) $\int_0^t (Y_s - Y_s^{(n)})^2 \, d\langle M \rangle_s \overset{a.s.}{=} \int_0^t \underbrace{(Y_s - Y_s^{(n)})^2 \frac{d\langle M \rangle_s}{ds}}_{\leq 4K^2 \frac{d\langle M \rangle_s}{ds} \in L^1([0,t])} ds \xrightarrow[n\to\infty]{DCT} 0$ a.s.

$\text{LHS} \leq 4K^2 \langle M \rangle_t \in L^1$ so DCT:

$$E\left(\int_0^t (Y_s - Y_s^{(n)})^2 \, d\langle M \rangle_s \right) \xrightarrow[n\to\infty]{} 0.$$

For $Y \in V_M$ unbounded, set $Y_t^{(n)} := Y_t 1_{|Y_t| \le n}$. Then by previous argument $Y_t^{(n)} \in \overline{V_0}^{\|\cdot\|_M}$.

Now $E\left(\int_0^t (Y_s - Y_s^{(n)})^2 d\langle M \rangle_s \right) = E\left(\int_0^t Y_s^2 1_{|Y_s| > n} \, d\langle M \rangle_s \right) \xrightarrow[n \to \infty]{DCT} 0.$ $\boxtimes$

In general, we have to assume a bit more measurability from Y.

Def Given a filtration $\{\mathcal{F}_t\}_{t \ge 0}$ on $(\Omega, \mathcal{F})$, a stoch process $\{X_t : t \ge 0\}$ is progressively measurable if $\forall t \ge 0 \ \forall B \in \mathcal{B}(\mathbb{R})$: $\{(\omega, s) \in \Omega \times [0,t] : X_s(\omega) \in B\} \in \mathcal{F}_t \otimes \mathcal{B}([0,t])$

Notes • Progressive meas $\Rightarrow$ adapted & jointly meas.

$\Omega \times \{t\} \in \mathcal{F}_t \otimes \mathcal{B}([0,t])$ $\quad$ $\mathcal{F}_t \subseteq \mathcal{F}$

$\Leftarrow$ fails in general, but counterexamples are contrived

• LC processes are progressively measurable.

Lemma Let T be stopping time for $\{\mathcal{F}_t\}_{t \ge 0}$. Then T is measurable w.r.t.

$$\mathcal{F}_T := \{A \in \mathcal{F} : (\forall t \ge 0 : A \cap \{T \le t\} \in \mathcal{F}_t)\}$$

If X is progressively meas. and T is finite then X_T is $\mathcal{F}_T$-measurable.

The key reason!

Pf (1) Fix $u, t \geq 0$. Then $\{T \leq u\} \cap \{T \leq t\} = \{T \leq t \wedge u\} \in \mathcal{F}_{t \wedge u} \subseteq \mathcal{F}_t$

So $\forall u \geq 0$: $\{T \leq u\} \in \mathcal{F}_T$. So $\sigma(T) \subseteq \mathcal{F}_T$.

(2) X progressively meas $\Rightarrow$ $(\omega, s) \longmapsto X_s(\omega)$ is $\mathcal{F}_t \otimes \mathcal{B}([0,t]) / \mathcal{B}(\mathbb{R})$ - meas.

$(\Omega \times [0,t] \to \mathbb{R})$

T random variable $\Rightarrow$ $\omega \longmapsto (\omega, T(\omega) \wedge t)$ is $\mathcal{F}_t / \mathcal{F}_t \otimes \mathcal{B}([0,t])$ - meas.

So $\omega \longmapsto X_{T(\omega) \wedge t}(\omega)$ is $\mathcal{F}_t / \mathcal{B}(\mathbb{R})$ - meas. This means: $\forall B \in \mathcal{B}(\mathbb{R})$: $\{X_{T \wedge t} \in B\} \in \mathcal{F}_t$

But $\{X_T \in B\} \cap \{T \leq t\} = \{X_{T \wedge t} \in B\} \cap \{T \leq t\} \overset{T \text{ stop. time}}{\in} \mathcal{F}_t$

So $\{X_T \in B\} \in \mathcal{F}_T$. $\boxtimes$

Prop $\forall M \in \mathcal{M}_2^{cont}$ $\forall Y \in \mathcal{V}_M$: Y progressively measurable $\Rightarrow$ $Y \in \overline{\mathcal{V}_0}^{[\cdot]_M}$.

Pf (idea) Invoke random time change & only then approximate by simple processes.

For $u \geq 0$ let $T(u) := \inf\{t \geq 0 : t + \langle M \rangle_t \geq u\}$ ensures existence of inverse

So $u \mapsto T(u)$ is continuous, strictly increasing with $T(0) = 0$ and

$$\forall u \geq 0: \quad T(u) + \langle M \rangle_{T(u)} = u$$

Moreover, $T(u)$ is stopping time and $\{\mathcal{F}_{T(u)}\}_{u \geq 0}$ is filtration.

Assume WLOG $Y \in \mathcal{V}_M$ is bounded, $\sup_{t\ge 0}|Y_t| \le K$.

There exist $\{Y^{(n)}\}_{n\in\mathbb{N}} \in \mathcal{V}_0^{\mathbb{N}}$ s.t. $E\left(\int_0^t \left(Y_{T(u)} - Y_u^{(n)}\right)^2 du\right) \xrightarrow[n\to\infty]{} 0$

w.r.t. $\{\mathcal{F}_{T(u)}\}_{u\ge 0}$

Now set $\tilde{Y}_t^{(n)} := Y^{(n)}_{t+\langle M\rangle_t}$. Then $\tilde{Y}^{(n)}_{T(u)} = Y^{(n)}_u$. Then

$$E\left(\int_0^t \left(Y_s - \tilde{Y}_s^{(n)}\right)^2 d\langle M\rangle_s\right) \le E\left(\int_0^t \left(Y_s - \tilde{Y}_s^{(n)}\right)^2 d(\langle M\rangle_s + s)\right)$$

$$\underset{s=T(u)}{=} E\left(\int_0^{t+\langle M\rangle_t} \left(Y_{T(u)} - \tilde{Y}^{(n)}_{T(u)}\right)^2 du\right)$$

$$\le E\left(\int_0^{t+a} \left(Y_{T(u)} - Y_u^{(n)}\right)^2 du\right) + 4K^2 E\left((t+\langle M\rangle_t) 1_{\langle M\rangle_t > a}\right) \xrightarrow[\substack{n\to\infty \\ a\to\infty}]{} 0.$$

Issue left: $\{\tilde{Y}_s^{(n)} : s \ge 0\}$ is not simple but of the form

$$\tilde{Y}_t^{(n)} = 1_{\{0\}}(t) Z_0 + \sum_{i=1}^{n} Z_i 1_{(T(t_{i-1}), T(t_i)]}(t)$$

Proceed by discretization of $T(\cdot)$. See the notes. $\boxtimes$

Corollary: $\forall M \in \mathcal{M}_2^{cont}$ $\forall Y \in \mathcal{V}_M$ TFAE:

(1) $Y \in \overline{\mathcal{V}_0}^{[\cdot]_M}$

(2) $\exists \hat{Y} \in \mathcal{V}_M$ progressively meas. s.t. $\int_0^\infty 1_{\{Y_t \neq \hat{Y}_t\}} d\langle M\rangle_t = 0$ a.s.

<u>Pf of Corollary:</u> (2) $\Rightarrow$ (1) by Prop. above

(1) $\Rightarrow$ (2) : Suppose $Y \in \overline{\mathcal{V}_0}^{[\cdot]_M}$. Let $\{Y^{(n)}\}_{n \in \mathbb{N}} \in \mathcal{V}_0^{\mathbb{N}}$ be s.t.

$$\int_0^\infty \mathbb{1}_{\{\limsup_{n\to\infty} |Y_t^{(n)} - Y_t| > 0\}} \, d\langle M \rangle_t = 0 \quad \text{a.s.}$$

Set $\tilde{Y}_t := \left(\limsup_{n\to\infty} Y_t^{(n)}\right) \mathbb{1}_{|\limsup_{n\to\infty} Y_t^{(n)}| < \infty}$. Then $\tilde{Y}$ is progressively meas.

and $\int_0^\infty \mathbb{1}_{\{Y_t \neq \tilde{Y}_t\}} \, d\langle M \rangle_s = 0$ a.s.