Construct quadratic variation & move to stoch. integral w.r.t. cont. local martingales

last time: M martingale with $\sup_{t\geq 0} |M_t| \leq K$ a.s. then for $\|\widetilde{\Pi}\| \leq \|\Pi\|$.

$$E\left(\left| V_t^{(2)}(M,\Pi) - V_t^{(2)}(M,\widetilde{\Pi}) \right|^2 \right) \leq 112\, K^2 \sqrt{E\left(\operatorname{osc}_M([0,t],\|\Pi\|)^4 \right)}$$

Lemma The Theorem holds for bounded continuous martingale: (Assuming $\mathcal{F}_0 \supseteq P$-null sets)

That is: $\sup_{t\geq 0} |M_t| \leq K$ a.s. implies existence of $\{\langle M\rangle_t : t \geq 0\}$ that is adapted, continuous, non-decreasing with $\langle M\rangle_0 = 0$ and $\{M_t^2 - \langle M\rangle_t : t \geq 0\}$ is a martingale

Moreover, $E\langle M\rangle_t \leq 4K^2$.

Pf: Idea: define $\langle M\rangle_t$ as L^2-limit of $V_t^{(2)}(M,\Pi)$ as $\|\Pi\| \to 0$.

Convention: $\Pi = \{0 = t_0 < t_1 < \dots < t_n\}$

$V_t^{(2)}(M,\Pi)$ defined by inserting t into Π and dropping $t_i \geq t$.

Note $t \longmapsto V_t^{(2)}(M,\Pi)$ is continuous

But: Not $\uparrow$ in general

Note $0 \leq s \leq u \leq t$ then:

$$E\left((M_t - M_s)^2 \,\middle|\, \mathcal{F}_u \right) = E\left((M_t - M_u)^2 \,\middle|\, \mathcal{F}_u \right) + (M_u - M_s)^2 \quad \text{a.s.}$$

If $0 \le u \le t$: $E\left(V_t^{(2)}(M,\Pi) \middle| \mathcal{F}_u \right) = V_u^{(2)}(M,\Pi) + E\left((M_t - M_u)^2 \middle| \mathcal{F}_u \right)$

$ = E(M_t^2 | \mathcal{F}_u) - M_u^2$

Hence: $\{ V_t^{(2)}(M,\Pi) - M_t^2 : t \ge 0 \}$ is cont. martingale.

Denote: $A_t^{\Pi} := V_t^{(2)}(M,\Pi)$. Then

$\forall \Pi, \tilde{\Pi}$: $\{ A_t^{\Pi} - A_t^{\tilde{\Pi}} : t \ge 0 \}$ is cont L^2-martingale.

Doob's L^2-ineq: $E\left(\sup_{s \le t} |A_s^{\Pi} - A_s^{\tilde{\Pi}}|^2 \right) \le 4 E\left(|A_t^{\Pi} - A_t^{\tilde{\Pi}}|^2 \right)$

Now find $\{\Pi_n\}$ s.t. $\|\Pi_n\| \downarrow 0$ and

$\forall n \ge 1$: $E\left(\mathrm{osc}_M([0,n], \|\Pi_n\|)^4 \right) \le 64^{-n}$

Then Lemma from last time:

$$P\left(\sup_{s \le n} |A_s^{\Pi_n} - A_s^{\Pi_{n+1}}| > 2^{-n} \right) \overset{\text{chebyshev + Doob } L^2}{\le} 4\left(\frac{1}{2^{-n}}\right)^2 E\left(|A_n^{\Pi_n} - A_n^{\Pi_{n+1}}|^2 \right)$$

$$\le 4 \cdot 112 \left(\frac{1}{2^{-n}}\right)^2 \sqrt{64^{-n}} = 448 \cdot 2^{-n}$$

Borel-Cantelli: $\Omega^* := \Omega \setminus \{ \sup_{s \le n} |A_s^{\Pi_n} - A_s^{\Pi_{n+1}}| > 2^{-n} \text{ i.o.} \}$ obeys $P(\Omega^*) = 1$

Define

$$A_t := \begin{cases} \lim_{n \to \infty} A_t^{\Pi_n} & \text{on } \Omega^* \\ 0 & \text{on } \Omega \setminus \Omega^* \end{cases}$$

Then A is continuous, adapted with $\forall t \ge 0$: $V_t^{(2)}(M,\Pi) \xrightarrow[\|\Pi\| \to 0]{L^2} A_t$

So $\{M_t^2 - A_t : t \geq 0\}$ is a martingale and $A_t \geq 0$ a.s. with $E(A_t) \leq (2K)^2$

We don't know whether A is non-decreasing.

Since for Π a partition containing $s \geq 0$ and any $t \geq s$:

$$A_t^{\Pi} - A_s^{\Pi} = V_t^{(2)}(M,\Pi) - V_s^{(2)}(M,\Pi) \geq 0$$

Hence by L^2-limit: $\forall t \geq s \geq 0$: $P(A_t \geq A_s) = 1$.

Define $\Omega^{**} := \bigcap_{\substack{s,t \in \mathbb{Q} \\ 0 \leq s < t}} \{A_s \leq A_t\}$. Then $P(\Omega^{**}) = 1$

and $\langle M \rangle_t := \begin{cases} A_t & \text{on } \Omega^{**} \\ 0 & \text{on } \Omega \setminus \Omega^{**} \end{cases}$ has required properties $\boxtimes$

<u>Pf of Thm:</u> Let M = cont. local martingale.

Set $\tau_K := \inf\{t \geq 0 : |M_t| \geq K\}$.

Then $M_t^K := M_{\tau_K \wedge t}$ obeys $\sup_{t \geq 0} |M_t^K| \leq K$.

So there exists $\langle M^K \rangle$ s.t. $\{M_t^K - \langle M^K \rangle_t : t \geq 0\}$ is a cont. martingale.

Then $\forall L \geq K$: $\{\langle M^L \rangle_{\tau_K \wedge t} - \langle M^K \rangle_{\tau_K \wedge t} : t \geq 0\}$ is cont. martingale

Uniqueness lemma: $\forall L \geq K$: $P(\underbrace{\forall t \leq \tau_K : \langle M^L \rangle_t = \langle M^K \rangle_t}_{=: \Omega_{L,K}}) = 1$

Set $\Omega_0 := \left(\bigcap_{k \geq 1} \bigcap_{l \geq k} \Omega_{lk} \right) \cap \{ \tau_k \to \infty \}$.

Let $\langle M \rangle_t := \begin{cases} \lim_{k \to \infty} \langle M^k \rangle_t & \text{on } \Omega_0 \\ 0 & \text{else} \end{cases}$

Since $\langle M \rangle_t = \langle M^k \rangle_t$ for $t \leq \tau_k$ (the limit exists) and $\{ M^2_{\tau_k \wedge t} - \langle M \rangle_{\tau_k \wedge t} : t \geq 0 \}$ is a martingale

Hence, $\{ M_t^2 - \langle M \rangle_t : t \geq 0 \}$ is local martingale. $\boxtimes$

Stoch. integral w.r.t. cont. local martingales

$Y \in \mathcal{V}_0$ with $Y_t = 1_{\{0\}}(t) Z_0 + \sum_{i=1}^n Z_i 1_{(t_{i-1}, t_i]}(t)$ and $\{M_t : t \geq 0\}$ given, set

$$\int_0^t Y_s \, dM_s := \sum_{i=1}^n Z_i \left(M_{t_i \wedge t} - M_{t_{i-1} \wedge t} \right)$$

Same regardless of representation of Y.

Notation: $\mathcal{M} = \{M : \text{martingale}\}$

$\mathcal{M}_2 = \{M : L^2\text{-martingale}\}$

$\mathcal{M}_2^{\text{cont}} = \{M : \text{continuous } L^2\text{-martingale}\}$.

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Lemma $\forall Y \in \mathcal{V}_0\ \forall M \in \mathcal{M}_2^{\text{cont}}$:

- $\{\int_0^t Y_s \, dM_s : t \geq 0\} \in \mathcal{M}_2$
- $\forall t \geq 0$: $E\left(\left[\int_0^t Y_s \, dM_s\right]^2\right) = E\left(\int_0^t Y_s^2 \, d\langle M\rangle_s\right)$
- $\left\{\left(\int_0^t Y_s \, dM_s\right)^2 - \int_0^t Y_s^2 \, d\langle M\rangle_s\right\} \in \mathcal{M}$.

Pf Only 3rd part: Let $0 \leq u \leq t$. Then assuming u is part of $\{t_i\}_{i=1}^n$, $u = t_j$

$$E\left(\left(\int_0^t Y_s \, dM_s\right)^2 \Big| \mathcal{F}_u\right) = \left(\int_0^u Y_s \, dM_s\right)^2 + E\left(\left(\sum_{i=j+1}^n Z_i (M_{t_{i \wedge t}} - M_{t_{i-1} \wedge t})\right)^2 \Big| \mathcal{F}_u\right)$$

$$\hat{E}\left((\cdots)^2 | \mathcal{F}_{t_{i-1}}\right)$$

$$= \left(\int_0^u Y_s \, dM_s\right)^2 + E\left(\underbrace{\int_u^t Y_s^2 \, d\langle M\rangle_s}_{\int_0^t Y_s^2 d\langle M\rangle_s - \int_0^u Y_s^2 d\langle M\rangle_s} \Big| \mathcal{F}_u\right)$$

gives 3rd part. $\boxtimes$

Def: $\mathcal{V}_M = \{ Y : \text{jointly meas} \wedge \text{adapted} \wedge \forall t \geq 0 : E\left(\int_0^t Y_s^2 \, d\langle M\rangle_s\right) < \infty \}$

$[\![Y - \tilde{Y}]\!]_M := \sum_{n \geq 1} 2^{-n} \max\left\{ 1, \sqrt{\int_0^n (Y_s - \tilde{Y}_s)^2 \, d\langle M\rangle_s} \right\}$

$\overline{\mathcal{V}}^{[\![\cdot]\!]_M} = \{ Y \in \mathcal{V}_M : \left(\exists \{Y_n\}_{n\in\mathbb{N}} \in \mathcal{V}_0^{\mathbb{N}} : [\![Y_n - Y]\!]_M \xrightarrow[n\to\infty]{} 0\right) \}$

Lemma The previous Lemma extends to all $Y \in \overline{\mathcal{V}_0}^{[\![\cdot]\!]_M}$.

with $\int_0^t Y_s \, dM_s := L^2\text{-limit of } \int_0^t Y_s^{(n)} dM_s$ for $Y^{(n)} \longrightarrow Y$

Lemma $\forall Y \in \overline{\mathcal{V}_0}^{[\![\cdot]\!]_M} \; \exists \{ I_t : t \geq 0 \} \in \mathcal{M}_2^{\text{cont}}$ s.t.

$\forall t \geq 0 : P\left(I_t = \int_0^t Y_s \, dM_s \right) = 1$

Moreover,

$\forall t \geq 0 : \langle I \rangle_t = \int_0^t Y_s^2 \, d\langle M\rangle_s$ (up to indistinguishability)