Quadratic variation of continuous local martingales

Last time: Loc. martingale: $\{M_t : t \geq 0\}$ s.t. $\exists T_n \to \infty$ stopping times for which $\forall n \geq 1$: $\{M_{T_n \wedge t} : t \geq 0\}$ is martingale

Ex $M_t = \int_0^t Y_s \, dB_s$, $Y \in \mathcal{V}^{loc}$, ...

Thm Assume $\mathcal{F}_0 \supseteq \{P\text{-null sets}\}$, let M = cont. loc. martingale. Then $\exists \{\langle M \rangle_t : t \geq 0\}$ = non-decreasing, cont., adapted process with $\langle M \rangle_0 = 0$ s.t. $\{M_t^2 - \langle M \rangle_t : t \geq 0\}$ is cont. loc. martingale

Moreover, $\langle M \rangle$ is unique up to indistinguishability. $\left(P(\forall t \geq 0 : X_t = \tilde{X}_t) = 1 \right)$

| Ex of Doob-Meyer decomposition! |

Pf of uniqueness:

Lemma Let M be a continuous local martingale s.t. $V_t^{(1)}(M) := \sup_{\Pi} V_t^{(1)}(M, \Pi) < \infty$ a.s. for some $t \geq 0$ we have

Then $P(\forall s \leq t : M_s = M_0) = 1$

Pf: Set $T_n := \inf\{s \geq 0 : |M_s| \geq n \vee V_s^{(1)}(M) \geq n\}$

Let $\tilde{M}_s := M_{T_n \wedge s}$. Then $\sup_{s \geq 0} |\tilde{M}_s| \leq n \wedge \sup_{s \geq 0} V_s^{(1)}(\tilde{M}) \leq n$

Now key identity: Let Π = partition of $[0,s]$ $\quad (s \le t)$

$$E\big((\tilde M_s - \tilde M_0)^2\big) = E\big(V_s^{(2)}(\tilde M, \Pi)\big)$$
$$\le E\big(\operatorname{osc}_{\tilde M}([0,t], \|\Pi\|)\, V_t^{(1)}(\tilde M, \Pi)\big)$$
$$\le n\, E\Big(\underbrace{\operatorname{osc}_{\tilde M}([0,t], \|\Pi\|)}_{\substack{\to 0 \text{ as } \|\Pi\| \to 0 \\ \le 2n \text{ by } \|\tilde M\|_\infty \le n}}\Big) \xrightarrow[\|\Pi\| \to 0]{BCT} 0.$$

So $P(\tilde M_s = \tilde M_0) = 1 \quad \forall s \ge 0$

So $P(\forall s \in [0, T_n \wedge t] \cap \mathbb{Q} : M_s = M_0) = 1$

Continuity of M: allows to drop containment in $\mathbb{Q}$.

Assumptions: $P(T_n \le t) \xrightarrow[n \to \infty]{} 0$ $\quad \boxtimes$

<u>Pf of uniqueness</u>: if $A, \tilde A$ are cont, non-decreasing, adapted with $A_0 = \tilde A_0 = 0$ s.t. $\{M_t^2 - A_t : t \ge 0\}$ and $\{M_t^2 - \tilde A_t : t \ge 0\}$ are loc. martingales then also $\{A_t - \tilde A_t : t \ge 0\}$ is cont. loc. martingale with $V_t^{(1)}(A - \tilde A) < \infty$ a.s. $\forall t \ge 0$.

So a.s.: $\forall t \ge 0$: $A_t - \tilde A_t = A_0 - \tilde A_0 = 0$ $\quad \boxtimes$

<u>Cor</u> For $M_t = \int_0^t Y_s \, dB_s$ with $Y \in \mathcal{V}^{loc}$:

$$\langle M \rangle_t = \int_0^t Y_s^2 \, ds$$

on $\{\forall t \ge 0 : \int_0^t Y_s^2 ds < \infty\}$ and $\langle M \rangle_t = 0$ elsewhere.

<u>Pf</u> For $Y \in \mathcal{V}$, we know $\{(\int_0^t Y_s dB_s)^2 - \int_0^t Y_s^2 ds : t \ge 0\}$ is cont. martingale. For $Y \in \mathcal{V}^{loc}$: localize. $\boxtimes$

Cor For U, Y and $\tilde{U}, \tilde{Y}$ s.t. integrals make sense:

$$\forall u \le t: \int_0^u U_s\,ds + \int_0^u Y_s\,dB_s = \int_0^u \tilde{U}_s\,ds + \int_0^u \tilde{Y}_s\,dB_s$$

implies
$$\mathrm{Leb}\left(s \in [0,t] : U_s \neq \tilde{U}_s \vee Y_s \neq \tilde{Y}_s \right) = 0 \text{ a.s.}$$

Pf $\int_0^u (Y_s - \tilde{Y}_s)\,dB_s = \int_0^u (\tilde{U}_s - U_s)\,ds$

LHS = cont. loc. martingale, RHS is $BV([0,t])$

Hence $\forall u \le t: \int_0^u (\tilde{U}_s - U_s)\,ds = 0 \;\wedge\; \int_0^t (Y_s - \tilde{Y}_s)^2\,ds = 0$

Apply Lebesgue's differentiation. $\boxtimes$

Proof of existence **Idea:** Construct $\langle M \rangle_t$ as L^2-limit of $V_t^{(2)}(M, \Pi)$ as $\|\Pi\| \to 0$.

Lemma Assume $\exists K > 0$: $\sup_{s \le t} |M_s| \le K$ a.s. Then for any partitions $\Pi, \tilde{\Pi}$ of $[0,t]$ with $\|\tilde{\Pi}\| \le \|\Pi\|$:

$$E\left(\left| V_t^{(2)}(M, \Pi) - V^{(2)}(M, \tilde{\Pi}) \right|^2 \right) \le 112\, K^2 \left[E\left(\mathrm{osc}_M([0,t], \|\Pi\|)^4 \right) \right]^{1/2}$$

Pf Assume first $\tilde{\Pi}$ refines Π. Write

$$\Pi = \{0 = t_{1,0} < t_{2,0} < \dots < t_{n,0} < t_{n+1,0} = t\}$$

$$\tilde{\Pi} = \{0 = t_{1,0} < t_{1,1} < \dots < t_{1,m_1} = t_{2,0} < \dots < t_{n,0} < t_{n,1} \dots < t_{n,m_n} = t_{n+1,0} = t\}$$

Denote $\quad Z_{i,j} = M_{t_{i,j}} - M_{t_{i,j-1}} \quad i = 1, \dots, n \,;\; j = 1, \dots, m_i$

With this:

$$V_t^{(2)}(M, \tilde{\Pi}) = \sum_{i=1}^{n} \sum_{j=1}^{m_i} Z_{i,j}^2$$

$$V_t^{(2)}(M, \Pi) = \sum_{i=1}^{n} \Big(\sum_{j=1}^{m_i} Z_{i,j} \Big)^2$$

$$\int_0 \quad (*) = V_t^{(2)}(M, \Pi) - V_t^{(2)}(M, \tilde{\Pi}) = 2 \sum_{i=1}^{n} \sum_{1 \le j < j' \le m_i} Z_{i,j} Z_{i,j'}$$

$$E(|(*)|^2) = 4 \sum_{i,i'=1}^{n} \sum_{1 \le j < j' \le m_i} \sum_{1 \le k < k' \le m_{i'}} E\big(Z_{i,j} Z_{i,j'} Z_{i',k} Z_{i',k'} \big)$$

$$= 4 \sum_{i=1}^{n} \sum_{1 \le j' \le m_i} \sum_{1 \le j,k < j'} E\big(Z_{i,j} Z_{i,k} (Z_{i,j'})^2 \big)$$

$$= 4 \sum_{i=1}^{n} \sum_{1 \le j' \le m_i} E\Big((M_{t_{i,j'-1}} - M_{t_{i,0}})^2 (M_{t_{i,j'}} - M_{t_{i,j'-1}})^2 \Big)$$

$$\le 4 E\Big(\operatorname{osc}_M([0,t], \|\Pi\|)^2 \, V_t^{(2)}(M, \tilde{\Pi}) \Big)$$

Now Cauchy-Schwarz, need to bound $E(V_t^{(2)}(M,\tilde{\Pi})^2)$

Rewrite $\tilde{\Pi} = \{0 = s_0 < \dots < s_r = t\}$. Then

$$E(V_t^{(2)}(M,\tilde{\Pi})^2) = E(V^{(4)}(M,\tilde{\Pi})) + 2\sum_{1\le i<j\le r} E((M_{s_i}-M_{s_{i-1}})^2(M_{s_j}-M_{s_{j-1}})^2)$$

$$= E(V^{(4)}(M,\tilde{\Pi})) + 2\sum_{1\le i<r} E((M_{s_i}-M_{s_{i-1}})^2(M_t-M_{s_i})^2)$$

$$\le 4K^2 E(V_t^{(2)}(M,\tilde{\Pi})) + 8K^2 E(V_t^{(2)}(M,\tilde{\Pi}))$$

$$= 12K^2 E((M_t-M_0)^2) \le 48K^2 \le (7K)^2$$

Now put together to get:

$$E(|(*)|^2) \le 4\cdot 7K \left[E(\mathrm{osc}_M([0,t],|\Pi|)^4)\right]^{1/2}.$$

This proves the claim for nested partitions. $\boxtimes$

For general case, apply Δ-inequality.