Itô formula for diffusions

last time: diffusion $dX_t = U_t dt + Y_t dB_t$ (unique)

$$\int_0^t Z_s dX_s := \int_0^t Z_s U_s ds + \int_0^t Z_s Y_s dB_s$$

Itô formula: $\forall f \in C^2(\mathbb{R})\ \forall t \geq 0$:

$$f(X_t) = f(X_0) + \int_0^t f'(X_s) dX_s + \frac{1}{2}\int_0^t f''(X_s) d\langle X\rangle_s$$

where $\langle X\rangle_t := \int_0^t Y_s^2 ds$

Proved for U, Y simple.

Proof for general U, Y: Let $U^{(n)}, Y^{(n)}$ be simple s.t.

$$\int_0^t |U_s^{(n)} - U_s| ds \xrightarrow[n\to\infty]{P} 0 \wedge \int_0^t |Y_s^{(n)} - Y_s|^2 ds \xrightarrow[n\to\infty]{P} 0$$

define $X_t^{(n)} := X_0 + \int_0^t U_s^{(n)} ds + \int_0^t Y_s^{(n)} dB_s$

$$\boxed{\text{Lemma} \quad \sup_{s\leq t} |X_s^{(n)} - X_s| \xrightarrow[n\to\infty]{P} 0}$$

Pf of Lemma: $\sup_{s\le t}|X_s^{(n)}-X_s| \le \underbrace{\int_0^t |U_s^{(n)}-U_s|\,ds}_{\to 0} + \sup_{u\le t}\left|\int_0^u (Y_s^{(n)}-Y_s)\,dB_s\right|$

Let $T_n := \inf\{u\ge 0 : \int_0^u (Y_s^{(n)}-Y_s)^2 ds \ge 1\}$, assumptions: $T_n \xrightarrow[n\to\infty]{} \infty$

Then $P\left(\sup_{u\le t}\left|\int_0^u (Y_s^{(n)}-Y_s)\,dB_s\right|>\varepsilon\right)$

$\le P(T_n\le t) + P\left(\sup_{u\le t}\left|\int_0^{u\wedge T_n}(Y_s^{(n)}-Y_s)\,dB_s\right|>\varepsilon\right)$

$\overset{\text{Doob } L^2,\ \text{Doob-Itô}}{\le} P(T_n\le t) + \frac{1}{\varepsilon^2}E\left(\int_0^{T_n\wedge t}(Y_s^{(n)}-Y_s)^2 ds\right)$

$\xrightarrow[n\to\infty]{} 0$ by assumptions $\boxtimes$

Lemma $\forall h\in C(\mathbb{R})\ \forall t\ge 0$: $\int_0^t h(X_s^{(n)})\,dX_s^{(n)} \longrightarrow \int_0^t h(X_s)\,dX_s$

Pf $\int_0^t |h(X_s^{(n)})U_s^{(n)} - h(X_s)U_s|\,ds$

$\le \sup_{p\le s\le t}|h(X_s^{(n)})-h(X_s)|\int_0^t |U_s^{(n)}|\,ds + \sup_{s\le t}|h(X_s)|\int_0^t |U_s^{(n)}-U_s|\,ds$

$\xrightarrow[n\to\infty]{} 0$ by ass. and Lemma

$\int_0^t |h(X_s^{(n)})Y_s^{(n)} - h(X_s)Y_s|^2 ds \overset{(a+b)^2\le 2a^2+2b^2}{\le} 2\left(\sup_{s\le t}|h(X_s^{(n)})-h(X_s)|^2\right)\int_0^t (Y_s^{(n)})^2 ds + 2\left(\sup|h(X_s)|^2\right)\int_0^t (Y_s^{(n)}-Y_s)^2 ds \longrightarrow 0$

So by Lemma last time: $\int_0^t h(X_s^{(n)})Y_s^{(n)}\,dB_s \xrightarrow[n\to\infty]{P} \int_0^t h(X_s)Y_s\,dB_s$ $\boxtimes$

Lemma $\forall h \in C(\mathbb{R})\ \forall t \geq 0$: $\int_0^t h(X_s^{(n)})\, d\langle X^{(n)}\rangle_s \xrightarrow[n\to\infty]{P} \int_0^t h(X_s)\, d\langle X\rangle_s$

Pf $d\langle X^{(n)}\rangle_s = (Y_s^{(n)})^2 ds$, ... proceed similarly. $\boxtimes$

Combining the lemmas we get

$$f(X_t) = \lim_{n\to\infty} f(X_t^{(n)})$$
$$\overset{P}{=} \lim_{n\to\infty} \left[f(X_0) + \int_0^t f'(X_s^{(n)})\, dX_s^{(n)} + \frac{1}{2}\int_0^t f''(X_s^{(n)})\, d\langle X^{(n)}\rangle_s \right]$$
$$\overset{P}{=} f(X_0) + \int_0^t f'(X_s)\, dX_s + \frac{1}{2}\int_0^t f''(X_s)\, d\langle X\rangle_s$$

Thm Suppose $B^{(1)} \ldots B^{(d)}$ are iid SBM adapted to common filtration $\{\mathcal{F}_t\}$ s.t. $\mathcal{F}_0$ contains all P-null sets. Assume diffusions $X^{(1)} \ldots X^{(m)}$ are given by

$$dX_t^{(i)} = U_t^{(i)} dt + \sum_{j=1}^{d} Y_t^{(i,j)} dB_t^{(j)}$$

Then $\forall f: \mathbb{R}_+ \times \mathbb{R}^m \to \mathbb{R}$ of type $C^1 \times C^2$, $\forall t \geq 0$:

$$f(t, X_t) = f(0, X_0) + \int_0^t \frac{\partial f}{\partial s}(s, X_s)\, ds + \sum_{i=1}^{m} \int_0^t \frac{\partial f}{\partial x_i}(s, X_s)\, U_s^{(i)} ds$$
$$+ \sum_{i=1}^{m}\sum_{j=1}^{d} \int_0^t \frac{\partial f}{\partial x_i}(s, X_s)\, Y_s^{(i,j)} dB_s^{(j)} + \frac{1}{2}\sum_{i,k=1}^{m}\sum_{j=1}^{d} \int_0^t \frac{\partial^2 f}{\partial x_i \partial x_k}(s, X_s)\, Y_s^{(i,j)} Y_s^{(k,j)} ds$$

Rules : $dt,\ dB_t^{(i)}$:

$$(dt)^2 = 0, \quad dt\, dB_t^{(i)} = 0, \quad dB_t^{(i)} dB_t^{(j)} = dt\, \delta_{i,j}$$

for iid SBM

Ex $(\underbrace{B_t^{(1)}, \dots B_t^{(d)}}_{\text{iid SBM}})$ = d-dimensional SBM.

$$R_t := \left[\sum_{i=1}^{d} (B_t^{(i)})^2 \right]^{1/2} \quad \dots \text{radial part}$$

$$= f(B_t^{(1)}, .., B_t^{(d)}) \text{ where } f(x_1 \dots x_d) := \left(\sum_{i=1}^{d} x_i^2 \right)^{1/2}$$

$X_t^{(i)} = B_t^{(i)}$

$U_t^{(i)} = 0, \ Y_t^{(i,j)} = \delta_{i,j}$

$$\frac{\partial f}{\partial x_k} = \frac{x_k}{f(x)}, \quad \frac{\partial^2 f}{\partial x_k^2} = \frac{1}{f(x)} - \frac{x_k^2}{f(x)^3}$$

then $$dR_t = \sum_{i=1}^{d} \frac{B_t^{(i)}}{R_t} dB_t^{(i)} + \frac{1}{2} \sum_{i=1}^{d} \left(\frac{1}{R_t} - \frac{(B_t^{(i)})^2}{R_t^3} \right) dt$$

$$= \frac{1}{R_t} \sum_{i=1}^{d} B_t^{(i)} dB_t^{(i)} + \frac{d-1}{2R_t} dt$$

will show R_t is a Bessel process.

Generalizing to semimartingales

recall martingale $\{M_t : t \geq 0\}$ is a adapted process s.t. $M_t \in L^1$ $\wedge$ $E(M_t | \mathcal{F}_s) = M_s$ a.s. $\forall t \geq s \geq 0$.

continuous if $t \mapsto M_t$ cont.

Def A local martingale is a process $\{M_t : t \geq 0\}$ that is adapted and for which there exist $\{T_n : n \geq 1\}$ s.t.

(1) $\forall n \geq 0$. T_n is stopping time $\wedge$ $\{M_{T_n \wedge t} : t \geq 0\}$ is a martingale

(2) $T_n \xrightarrow[n \to \infty]{} \infty$ a.s.

Ex • $M_t = \int_0^t Y_s \, dB_s$ $\quad Y \in V^{loc}$

• $M_t = B_{\sigma(t)}$ where $\sigma : \mathbb{R}_+ \to \mathbb{R}_+$ cont, $\uparrow$, $\sigma(0) = 0$

Thm Let M be loc. martingale w.r.t $\{\mathcal{F}_t\}_{t \geq 0}$ s.t. $\mathcal{F}_0 \supseteq \{P\text{-null sets}\}$.
Then $\exists$ $\{\langle M \rangle_t : t \geq 0\}$ cont., adapted, non-decreasing with $\langle M \rangle_0 = 0$ s.t.

$$\{M_t^2 - \langle M \rangle_t : t \geq 0\} \text{ is loc. martingale}$$

Any two such $\langle M \rangle$-processes are indistinguishable.