Properties of Itô integral, moving to diffusions

last time: Linearity of $Y \mapsto \int_0^t Y_s\,dB_s$

Additivity w.r.t. $[0,t]$ ... $\int_u^t Y_s\,dB_s := \int_0^t Y_s\,dB_s - \int_0^u Y_s\,dB_s$

Lemma: Let $Y \in \mathcal{V}^{loc}$, $u \geq 0$, $Z = \mathcal{F}_u$-meas. r.v.

Then $\{Z \mathbb{1}_{[u,\infty)}(s) Y_s ; s \geq 0\} \in \mathcal{V}^{loc}$ and

$$\forall t \geq u: \quad \int_0^t Z \mathbb{1}_{[u,\infty)}(s) Y_s\,dB_s = Z \int_u^t Y_s\,dB_s \quad \text{a.s.}$$

Pf: HW4

Then we showed: $\int_0^t Y_s\,dB_s = \int_0^t \tilde{Y}_s\,dB_s$ a.s. $\Leftarrow$ $\mathrm{Leb}(\{s \leq t : Y_s \neq \tilde{Y}_s\}) = 0$ a.s.

Lemma: Let $\{Y^{(n)}\}_{n\in\mathbb{N}} \in (\mathcal{V}^{loc})^{\mathbb{N}}$, $Y \in \mathcal{V}^{loc}$. Then

$$\forall t \geq 0: \int_0^t |Y_s^{(n)} - Y_s|^2\,ds \xrightarrow[n\to\infty]{P} 0 \Rightarrow \int_0^t Y_s^{(n)}\,dB_s \xrightarrow[n\to\infty]{P} \int_0^t Y_s\,dB_s$$

Pf: Localization (HW4)

This allows us to prove Itô formula $f(B_t) = f(B_0) + \int_0^t f'(B_s)\,dB_s + \frac{1}{2}\int_0^t f''(B_s)\,ds$ whenever $f \in C^2(\mathbb{R})$, in fact, for $f \in C^1(\mathbb{R})$ s.t. $f' \in AC(\mathbb{R})$ (FTC holds for f')

Diffusions

(Assume $\mathcal{F}_0$ contains all P-null sets)

No Make-up class tomorrow

With Itô integral, we have a large class of continuous stoch. processes $\{\int_0^t Y_s \, dB_s : t \geq 0\}$ $\forall Y \in \mathcal{V}^{loc}$

Def (Generalized diffusions) Given $(\Omega, \mathcal{F}, P)$ supporting SBM $\{B_t : t \geq 0\}$ and Br. filtration $\{\mathcal{F}_t\}_{t \geq 0}$, a process $\{X_t : t \geq 0\}$ is <u>generalized diffusion</u> if there exist $\{U_t : t \geq 0\}$, $\{Y_t : t \geq 0\}$ s.t.

(1) U, Y, X are adapted, jointly measurable

(2) $\forall t \geq 0$: $\int_0^t |U_s| \, ds < \infty \wedge \int_0^t Y_s^2 \, ds < \infty$ a.s.

(3) $\forall t \geq 0$: $X_t = X_0 + \int_0^t U_s \, ds + \int_0^t Y_s \, dB_s$ a.s.

(4) X is continuous

Remark
- Infinitesimal form: $dX_t = U_t \, dt + Y_t \, dB_t$
- $\int_0^t U_s \, ds$ = drift term, $\int_0^t Y_s \, dB_s$ = diffusive term
- "Proper" diffusion: $U_t = f(t, X_t)$, $Y_t = g(t, X_t)$
  (reflects on locality principles in physics)

Will now introduce integration w.r.t. diffusions

Def Let X be diffusion in the form $dX_t = U_t\,dt + Y_t\,dB_t$

Let $\{Z_t : t \geq 0\}$ be adapted, jointly measurable s.t.

$$\forall t \geq 0 : \quad \int_0^t |Z_s||U_s|\,ds < \infty \;\wedge\; \int_0^t Z_s^2 Y_s^2\,ds < \infty \quad \text{a.s.}$$

Then
$$\int_0^t Z_s\,dX_s := \int_0^t Z_s U_s\,ds + \int_0^t Z_s Y_s\,dB_s$$

Abbreviate:
$$\langle X \rangle_t := \int_0^t Y_s^2\,ds \qquad \left(d\langle X \rangle_t = Y_t^2\,dt \right)$$

(**Note** $\left\{ \left(\int_0^t Y_s\,dB_s \right)^2 - \langle X \rangle_t : t \geq 0 \right\}$ is a martingale).

Thm (Itô formula for diffusion) Let X be a diffusion.

Then $\forall f \in C^2(\mathbb{R})$:

$$\forall t \geq 0 : \quad f(X_t) = f(X_0) + \int_0^t f'(X_s)\,dX_s + \frac{1}{2} \int_0^t f''(X_s)\,d\langle X \rangle_s .$$

Stieltjes integral

Note $s \mapsto f''(X_s) \in C([0,\infty))$, $s \mapsto \langle X \rangle_s \in BV([0,t])$ $\Big\}$ Stieltjes integral exists.

<u>Pf for simple U & Y:</u> Assume U, Y are simple, take Π = partition of $[0,t]$ s.t. all partition points of U and Y are included.
Assume wlog that U, Y are RC (instead of LC). Now for $\Pi = \{t_i\}_{i=0}^m$:

$$f(X_t) - f(X_0) = \sum_{i=1}^m \left[f(X_{t_i}) - f(X_{t_{i-1}}) \right]$$

$$\overset{\text{Taylor}}{=} \sum_{i=1}^m f'(X_{t_{i-1}})(X_{t_i} - X_{t_{i-1}}) + \sum_{i=1}^n \int_0^1 ds\, f''(sX_{t_i} + (1-s)X_{t_{i-1}})(1-s)(X_{t_i} - X_{t_{i-1}})^2$$

<u>Note</u> $X_{t_i} - X_{t_{i-1}} = U_{t_{i-1}}(t_i - t_{i-1}) + Y_{t_{i-1}}(B_{t_i} - B_{t_{i-1}})$

$$1^{st} \text{ term} = \underbrace{\sum_{i=1}^m f'(X_{t_{i-1}}) U_{t_{i-1}}(t_i - t_{i-1})}_{\xrightarrow[\|\Pi\|\to 0]{} \int_0^t f'(X_s) U_s\, ds} + \sum_{i=1}^m f'(X_{t_{i-1}}) Y_{t_{i-1}} (B_{t_i} - B_{t_{i-1}}) \xrightarrow[\|\Pi\|\to 0]{P} \int_0^t f'(X_s) Y_s\, dB$$

because $f\circ X \in C(\mathbb{R}_+)$, Y piecewise constant

we used $\displaystyle\sum_{i=1}^m \int_{t_{i-1}}^{t_i} |f'(X_{t_{i-1}}) Y_{t_{i-1}} - f'(X_s) Y_s|^2 ds \le \sup_{s\le t} \|Y_s\|_\infty\, t\, \underbrace{\operatorname{osc}_{f\circ X}([0,t], \|\Pi\|)^2}_{\xrightarrow[\|\Pi\|\to 0]{} 0}$

$$
\begin{aligned}
\underline{2^{nd}\text{ term}} = & \frac{1}{2} \sum_{i=1}^m f''(X_{t_{i-1}}) Y_{t_{i-1}}^2 (B_{t_i} - B_{t_{i-1}})^2 && \dots J^{(1)}(\Pi) \\
& + \sum_{i=1}^m f''(X_{t_{i-1}}) Y_{t_{i-1}} U_{t_{i-1}} (B_{t_i} - B_{t_{i-1}})(t_i - t_{i-1}) && \dots J^{(2)}(\Pi) \\
& + \frac{1}{2} \sum_{i=1}^m f''(X_{t_{i-1}}) U_{t_{i-1}}^2 (t_i - t_{i-1})^2 && \dots J^{(3)}(\Pi) \\
& + \sum_{i=1}^m \int_0^1 ds\, (1-s) \left[f''(sX_{t_i} + (1-s)X_{t_{i-1}}) - f''(X_{t_{i-1}}) \right] (X_{t_i} - X_{t_{i-1}})^2 && \dots J^{(4)}(\Pi)
\end{aligned}
$$

$$J^{(1)}(\Pi) - \sum_{i=1}^{n} f''(X_{t_{i-1}}) Y_{t_{i-1}}^2 (t_i - t_{i-1})$$

$$= \sum_{i=1}^{n} f''(X_{t_{i-1}}) Y_{t_{i-1}}^2 \left[(B_{t_i} - B_{t_{i-1}})^2 - (t_i - t_{i-1}) \right]$$

This is martingale if truncate $f''(\cdot)$. So truncate and bound this using Chebyshev.

Remaining terms are shown to $\xrightarrow[\|\Pi\| \to 0]{} 0$ by oscilation bounds.

Since $\sum_{i=1}^{n} f''(X_{t_{i-1}}) Y_{t_{i-1}}^2 (t_i - t_{i-1}) \xrightarrow[\|\Pi\| \to 0]{} \int_0^t f''(X_s) Y_s^2 \, ds$

we have the claim for simple U, Y.