Localization & properties of Itô integral

__Last time:__ __Prop__ Assume $\mathcal{F}_0$ contains all P-null sets. Let $Y \in \mathcal{V}$ and let T = stopping time. ($\forall t \geq 0$: $\{T \leq t\} \in \mathcal{F}_t$)

Then $\forall t \geq 0$:
$$\int_0^{T\wedge t} Y_s \, dB_s := I_{T\wedge t} = \int_0^t Y_s \mathbb{1}_{T>s} \, dB_s \text{ a.s.}$$

where $\{I_t : t \geq 0\}$ is a cont. version of $\{\int_0^t Y_s dB_s : t \geq 0\}$.

__Pf__ Check for simple processes & discretized T & take limits.

__Thm__ Let $\mathcal{F}_0$ contain all P-null sets and $Y \in \mathcal{V}^{loc}$ where

$$\mathcal{V}^{loc} := \left\{ Y = \{Y_t : t \geq 0\} : \begin{array}{l} \text{jointly meas, adapted} \\ \forall t \geq 0: \int_0^t Y_s^2 ds < \infty \text{ a.s.} \end{array} \right\}$$

Let $\{T_N\}_{N \geq 1}$ be stopping times s.t. $T_N \uparrow \infty$ a.s. as $N \to \infty$ and

$$\forall N \geq 1: \{ Y_s \mathbb{1}_{T_N > s} : s \geq 0 \} \in \mathcal{V}.$$

Then $\forall N \geq M$ $\forall t \geq 0$: $\int_0^t Y_s \mathbb{1}_{T_N > s} \, dB_s = \int_0^t Y_s \mathbb{1}_{T_M > s} \, dB_s$ a.s. on $\{T_M > t\}$

and so $\int_0^t Y_s \, dB_s := \lim_{N\to\infty} \int_0^t Y_s \mathbb{1}_{T_N > s} \, dB_s$ exists & is finite a.s.

Pf: $N \geq M \Rightarrow T_N \geq T_M$ and by Prop:

$$\int_0^t Y_s 1_{T_M > s}\, dB_s = \int_0^t Y_s 1_{T_M > s} 1_{T_N > s}\, dB_s$$

$$\overset{\text{Prop}}{=} \int_0^{T_M \wedge t} Y_s 1_{T_N > s}\, dB_s \quad \text{a.s.}$$

$$= \int_0^t Y_s 1_{T_N > s}\, dB_s \quad \text{a.s. on } \{T_M > t\}$$

So limit exists on $\bigcup_{M \geq 1} \{T_M > t\} \supseteq \{T_M \to \infty\}$

$\square$

Remark: • canonical choice $T_N := \inf\{t \geq 0 : \int_0^t Y_s^2\, ds \geq N\}$

use this to define $\int_0^t Y_s\, dB_s$ $\forall Y \in \mathcal{V}^{loc}$.

- if $Y \in \mathcal{V}$ then this is the previous concept

$$\mathbb{E}\left(\left|\int Y_s 1_{T_N > t}\, dB_s - \int Y_s\, dB_s\right|^2\right) = \| Y 1_{T_N > \cdot} - Y \|^2_{L^2([0,t] \times \Omega)}$$

$$= \mathbb{E}\left(\int_0^t Y_s^2 1_{T_N \leq t}\, ds\right) \xrightarrow[N \to \infty]{DCT} 0$$

- independence of sequence of stopping times by "interlacing"

Remarks
- The technique of "working with stopped stoch. integrals" is referred to as localization. Often needed in applications.
- What if $Y \notin \mathcal{V}^{loc}$? E.g. if $T := \inf\{t \geq 0 : \int_0^t Y_s^2 \, ds = +\infty\}$ is finite? We find out:

$$\limsup_{t \uparrow T} \int_0^t Y_s \, dB_s = +\infty \wedge \liminf_{t \uparrow T} \int_0^t Y_s \, dB_s = -\infty.$$

I.e. there is no extension beyond T! (up to)

Properties of Itô integral ($\mathcal{F}_0$ contains all P-null sets)

Lemma (Linearity) $\forall Y, \tilde{Y} \in \mathcal{V}^{loc}$ $\forall \alpha, \beta \in \mathbb{R}$:

$$\alpha Y + \beta \tilde{Y} \in \mathcal{V}^{loc}$$

and $\forall t \geq 0$: $$\int_0^t (\alpha Y_s + \beta \tilde{Y}_s) \, dB_s = \alpha \int_0^t Y_s \, dB_s + \beta \int_0^t \tilde{Y}_s \, dB_s \quad \text{a.s.}$$

Pf: Joint meas & adapted immediate. $(a+b)^2 \le 2a^2 + 2b^2 \quad a,b>0$

$$\int_0^t (\alpha Y_s + \beta \tilde{Y}_s)^2 ds \le \alpha^2 \int_0^t Y_s^2 ds + \beta^2 \int_0^t \tilde{Y}_s^2 ds < \infty \quad \text{a.s.}$$

So $\alpha Y + \beta \tilde{Y} \in \mathcal{V}^{loc}$.

Additivity — already checked for $Y, \tilde{Y} \in \mathcal{V}$.

For $Y, \tilde{Y} \in \mathcal{V}^{loc}$ set $T_N := \inf\{t \ge 0 : \int_0^t (Y_s^2 + \tilde{Y}_s^2) ds \ge N\}$.

Then $Y_t^{(N)} = Y_t 1_{T_N > t}$, $\tilde{Y}_t^{(N)} := \tilde{Y}_t 1_{T_N > t}$ obey $Y^{(N)}, \tilde{Y}^{(N)} \in \mathcal{V}$.

So
$$\int_0^t (\alpha Y_s + \beta \tilde{Y}_s) 1_{T_N > s} dB_s = \int_0^t (\alpha Y_s^{(N)} + \beta \tilde{Y}_s^{(N)}) dB_s$$
$$= \alpha \int_0^t Y_s 1_{T_N > s} dB_s + \beta \int_0^{t} Y_s 1_{T_N > s} dB_s$$

Now take $N \to \infty$. $\boxtimes$

Additivity in integration domain:

Lemma Let $\mathcal{V}^{loc}$ be defined using SBM B and filtration $\{\mathcal{F}_t\}_{t\ge 0}$.
Pick $Y \in \mathcal{V}^{loc}$ and $u \ge 0$ and set
$$\tilde{B}_t := B_{t+u} - B_u, \quad \tilde{\mathcal{F}}_t := \mathcal{F}_{t+u}, \quad \tilde{Y}_t := Y_{t+u} \quad t \ge 0.$$
Then $\tilde{Y} \in \tilde{\mathcal{V}}^{loc}$ = defined using $\tilde{B}$ and $\{\tilde{\mathcal{F}}_t\}_{t\ge 0}$, and
$$\forall t \ge 0 \quad \int_0^{t+u} Y_s dB_s = \int_0^u Y_s dB_s + \int_0^t \tilde{Y}_s d\tilde{B}_s \quad \text{a.s.}$$

Pf Check for simple Y and take limits

Hence we can define

$$\forall t \geq u: \int_u^t Y_s \, dB_s := \int_0^t Y_s \, dB_s - \int_0^u Y_s \, dB_s$$

and treat it as an Itô integral.

Lemma (Conditional Itô isometry) $\forall t \geq u \geq 0 \ \forall Y \in \mathcal{V}$:

$$E\left(\left(\int_u^t Y_s \, dB_s \right)^2 \middle| \mathcal{F}_u \right) = E\left(\int_u^t Y_s^2 \, ds \middle| \mathcal{F}_u \right)$$

Pf. HW 4

Corollary $\forall Y \in \mathcal{V}$: Assuming a cont. version of $\{ \underbrace{\int_0^t Y_s \, dB_s}_{= M_t} : t \geq 0 \}$:

$$\left\{ \left(\int_0^t Y_s \, dB_s \right)^2 - \int_0^t Y_s^2 \, ds : t \geq 0 \right\}$$

is a continuous martingale (up to modification). (Doob-Meyer decomposition)

Pf. $t \geq u$:
$$E(M_t^2 | \mathcal{F}_u) = E((M_u + M_t - M_u)^2 | \mathcal{F}_u)$$
$$= M_u^2 + 2 M_u \underbrace{E(M_t - M_u | \mathcal{F}_u)}_{=0} + \underbrace{E((M_t - M_u)^2 | \mathcal{F}_u)}_{\text{Lemma } = E(\int_u^t Y_s^2 ds | \mathcal{F}_u)}$$

So $E\left(M_t^2 - \int_0^t Y_s^2 \, ds \middle| \mathcal{F}_u \right) = M_u^2 - \int_0^u Y_s^2 \, ds$. $\boxtimes$

Determinism:

Lemma Let $Y, \tilde{Y} \in V$, $t \geq 0$. Then

$$(1) \qquad \int_0^t Y_s \, dB_s = \int_0^t \tilde{Y}_s \, dB_s \quad \text{a.s.}$$

is equivalent to

$$(2) \qquad \mathrm{Leb}\left(\left\{ s \in [0,t] : Y_s^{(\omega)} \neq \tilde{Y}_s^{(\omega)} \right\} \right) = 0 \quad \text{a.s.}$$

Pf:

$$(1) \overset{\text{Itô iso}}{\Longleftrightarrow} E\left(\left| \int_0^t Y_s \, dB_s - \int_0^t \tilde{Y}_s \, dB_s \right|^2 \right) = 0$$

$$\overset{\text{Itô iso}}{\Longleftrightarrow} \| Y - \tilde{Y} \|_{L^2([0,t] \times \Omega)} = 0$$

$$\Longleftrightarrow (2)$$

Note
- True even if $Y, \tilde{Y} \in V^{loc}$
- True in form "equality holds a.s. on set of ω where (2) holds"