Localization & general Itô integral

last times: $Y \in \mathcal{V}$, $t \geq 0 \longmapsto \int_0^t Y_s \, dB_s$, admits cont. version if all P-null sets in $\mathcal{F}_0$.

$\boxed{\text{assumed throughout}}$

today: give up on square integrability w.r.t. P

$$\mathcal{V}^{loc} := \left\{ Y = \{Y_s : s \geq 0\} : \begin{array}{l} \text{jointly meas, adapted,} \\ \forall t \geq 0 : \int_0^t Y_s^2 \, ds < \infty \text{ a.s} \end{array} \right\}$$

Thm Let $Y \in \mathcal{V}^{loc}$ and for $M > 0$ set $T^{(M)} := \inf\{t \geq 0 : \int_0^t Y_s^2 \, ds \geq M\}$

Then (1) $\{ Y_t 1_{T^{(M)} > t} : t \geq 0 \} \in \mathcal{V}$

(2) $\forall t \geq 0 \; \forall N \geq M$: $\int_0^t Y_s 1_{T^{(N)} > s} \, dB_s = \int_0^t Y_s 1_{T^{(M)} > s} \, dB_s$ a.s. on $\{T^{(M)} > t\}$

In particular: $\forall t \geq 0$: $\int_0^t Y_s \, dB_s := \lim_{N \to \infty} \int_0^t Y_s 1_{T^{(N)} > s} \, dB_s$ exists and is finite a.s.

For $Y \in \mathcal{V}$, this coincides with previous def. of $\int_0^t Y_s \, dB_s$.

Def An $[0,\infty]$-valued r.v. T is a <u>stopping time</u> for filtration $\{\mathcal{F}_t\}_{t\ge 0}$ if $\forall t \ge 0$: $\{T \le t\} \in \mathcal{F}_t$

Lemma $T^{(M)}$ above is stopping time.

Pf: M(T) + $\mathcal{F}_0$ contains P-null sets $\Rightarrow \forall u \ge 0$: $\int_0^u Y_s^2\,ds$ is $\mathcal{F}_u$-meas.

Now
$$\{T^{(M)} \le t\} = \underbrace{\left\{\int_0^t Y_s^2\,ds \ge M\right\}}_{\in \mathcal{F}_t} \cup \left(\underbrace{\left\{\int_0^t Y_s^2\,ds < M\right\}}_{\in \mathcal{F}_t} \cap \bigcap_{n\ge 1} \underbrace{\left\{\int_0^{t+1/n} Y_s^2\,ds = +\infty\right\}}_{\in \mathcal{F}_0} \right)$$

So $\{T^{(M)} \le t\} \in \mathcal{F}_t$ $\boxtimes$

Cor $\forall Y \in \mathcal{V}^{loc}$ $\forall M > 0$: $\{Y_t 1_{T^{(M)} > t} : t \ge 0\} \in \mathcal{V}$.

Pf Joint meas & adaptedness by above lemma.

Since
$$\int_0^t (Y_s 1_{T^{(M)} > s})^2\,ds = \int_0^{T^{(M)}\wedge t} Y_s^2\,ds \overset{a.s.}{\le} M.$$

So $\mathbb{E}(\ldots) \le M$. $\boxtimes$

Prop (Stopped Itô integral) Assume $\mathcal{F}_0$ contains P-null sets.
Then $Y \in \mathcal{V}$ $\forall T =$ stopping time:

$$\{ Y_t 1_{T>t} : t \geq 0 \} \in \mathcal{V}$$

and writing $\{ I_t : t \geq 0\}$ for cont. version of $\{ \int_0^t Y_s \, dB_s : t \geq 0 \}$

$$\forall t \geq 0 : \int_0^{T \wedge t} Y_s \, dB_s := I_{T \wedge t} \overset{\text{claim}}{=} \int_0^t Y_s 1_{T>s} \, dB_s \quad \text{a.s.}$$

Pf structure:
- discretize T, replace Y by simple process
- take limits

Lemma Let $T =$ stopping time. For $N \in \mathbb{N}$ let $T_N := 2^{-N} \lceil 2^N T \rceil$.
Then T_N is a stopping time s.t. $T_N \downarrow T$ as $N \to \infty$.
In particular, $\{ 1_{T>t} : t \geq 0 \}$ is jointly meas.

Pf $T_N - 2^{-N} \leq T \leq T_N$, $T_N \downarrow$, $\{T_N \leq t\} = \{ T \leq \underbrace{2^{-N} \lfloor 2^N t \rfloor}_{\leq t} \}$

T_N discrete valued $\Rightarrow \{ 1_{T_N > t} : t \geq 0 \}$ is jointly meas.

taking limits we get claim. $\boxtimes$

Now pick $Y^{(n)} \in \mathcal{V}_0$ s.t. $[\![Y - Y^{(n)}]\!] \xrightarrow[n\to\infty]{} 0$.

<u>Lemma</u> $\forall n \geq 1 \ \forall N \geq 0$: $\{ Y^{(n)}_t 1_{T_N > t} : t \geq 0 \} \in \mathcal{V}$ (equivalent to $\mathcal{V}_0$-process)

and $\forall t \geq 0$: $\int_0^{T_N \wedge t} Y_s^{(n)} \, dB_s = \int_0^t Y_s^{(n)} 1_{T_N > s} \, dB_s$ a.s.

where integral on left is given by explicit formula.

<u>Pf</u> $\in \mathcal{V}$ by above reasoning.

Assume $Y_t^{(n)} = Z_0 1_{\{0\}}(t) + \sum_{i=1}^{m} Z_i 1_{(t_{i-1}, t_i]}(t)$

where $0 \leq t_0 < \dots < t_n$ are s.t. $\{t_0, \dots, t_m\} \supseteq \{ k 2^{-N} : k \in \mathbb{N} \} \cap \{ k 2^{-n} \leq t_m \}$

Define $R_t := \sum_{i=1}^{m} Z_i 1_{\{T_N > t_{i-1}\}} 1_{(t_{i-1}, t_i]}(t)$

<u>claim</u> $R \in \mathcal{V}_0$ and $\left| Y_t^{(n)} 1_{T_N > t} - R_t \right| \leq \sum_{i=1}^{m} \|Z_i\|_\infty 1_{\{t_i\}}(t)$

Now $\int_0^{T_N \wedge t} Y_s^{(n)} \, dB_s = \sum_{i=1}^{m} Z_i \left(B_{T_N \wedge t \wedge t_i} - B_{T_N \wedge t \wedge t_{i-1}} \right)$

$\int_0^t Y_s 1_{T_N > s} \, dB_s \overset{a.s.}{=} \int_0^t R_s \, dB_s = \sum_{i=1}^{m} Z_i 1_{T_N > t_{i-1}} \left(B_{t_i \wedge t} - B_{t_{i-1} \wedge t} \right)$

Since $T_N > t_{i-1} \Rightarrow T_N \geq t_i$ by ass. on $\{t_i\}_{i=0}^{m}$ the expressions are equal.

Taking limits:

Lemma $\lim_{N\to\infty} \limsup_{n\to\infty} E\left(\left|\int_0^t Y_s^{(n)} 1_{T_N>s}\, dB_s - \int_0^t Y_s 1_{T>s}\, dB_s\right|^2\right) = 0$

Pf $0 \le 1_{T^{(N)}>s} - 1_{T>s} \le 1_{T<s\le T_N}$

So $\| Y^{(n)} 1_{T_N>\cdot} - Y 1_{T>\cdot} \|_{L^2([0,t]\times\Omega)}$

$\le \| Y - Y^{(n)} \|_{L^2([0,t]\times\Omega)} + \underbrace{\| Y (1_{T_N>\cdot} - 1_{T>\cdot}) \|_{L^2([0,t]\times\Omega)}}_{\le \| Y 1_{T<\cdot\le T_N} \| \xrightarrow[N\to\infty]{DCT} 0}$

$\llcorner$

Lemma $\forall t\ge 0\ \forall N \ge 0$: $\int_0^{T_N\wedge t} Y_s^{(n)}\, dB_s \xrightarrow[n\to\infty]{P} \int_0^{T_N\wedge t} Y_s\, dB_s \ (:= I_{T_N\wedge t})$

Pf Both sides are cont. L^2-martingales evaluated at $T_N\wedge t$

So Doob's L^2-ineq:

$P(|LHS - RHS| > \varepsilon) \le P\left(\sup_{u\le t} \left|\int_0^u Y_s^{(n)}\, dB_s - I_u\right| > \varepsilon\right)$

$\underset{\text{Doob Itô}}{\le} \frac{1}{\varepsilon^2} \| Y^{(n)} - Y \|^2_{L^2([0,t]\times\Omega)} \xrightarrow[n\to\infty]{} 0$

$\llcorner$

Pf of Prop: Combine these lemmas