Stoch. Analysis

Kolmogorov's model $(\Omega, \mathcal{F}, P)$

$X : \Omega \to \mathcal{X}$ $\quad (\mathcal{X}, \Sigma)$ = meas. space

meas. $(\mathcal{F}/\Sigma)$ ... random variable

Notations:

$\{X \in B\} = X^{-1}(B) := \{\omega \in \Omega : X(\omega) \in B\}$

Facts: $\{\{X \in B\} : B \in \Sigma\}$ is σ-alg

if Σ is —"—

$\{B \subseteq \mathcal{X} : \{X \in B\} \in \mathcal{F}\}$ is σ-alg

if $\mathcal{F}$ is σ-alg.

Def Given meas. space $(\mathcal{X}, \Sigma)$ and $T \neq \emptyset$ set, an $\mathcal{X}$-valued stoch. process is $\{X_t : t \in T\}$ where X_t is $\mathcal{X}$-valued R.V. $\forall t \in T$

Ex $T := \mathbb{N}$ $\quad \{X_n : n \in \mathbb{N}\}$

$T := [0, \infty)$ stoch. process

t ... interpret as time

$T := \mathbb{R}^d$ random field

$T = \sigma$-alg. random measure

2 key examples

Def A homogeneous Poisson process is $\mathbb{N}$-valued $\{N_t : t \in [0,\infty)\}$ s.t.

1) $N_0 = 0$
2) $\forall n \geq 1\ \forall 0 = t_0 < t_1 < \dots < t_n$:

   $\{N_{t_i} - N_{t_{i-1}} : i = 1..n\}$ are indep.
3) $\forall 0 \leq s < t$: $N_t - N_s = \text{Poisson}(t-s)$
4) every realization of $t \to N_t$ is right continuous & has left limit

   RCLL / càdlàg

- models arrivals of customers to a queue
- inhomogeneous version

  3') ... $N_t - N_s = \text{Poisson}(G(t) - G(s))$

  $G : \mathbb{R}_+ \to \mathbb{R}_+$, $\uparrow$ RCLL

  e.g. $\{N_{G(t)} : t \in [0,\infty)\}$
- to be constructed later

(SBM)
motion

3 s.t.

$$= \int_A e^{-\frac{x^2}{2t}} \frac{dx}{\sqrt{2\pi t}}$$

<u>N.B.</u> Can replace 3) by requiring $X_t - X_s$ = infinite divisible

$E(e^{iuX_t})$ (Lévy-Khinchin formula)

$$= \exp\left\{ i\mu t\,{}_{u} - \frac{\sigma^2}{2} t\,{}_{u^2} - t \int_{\mathbb{R}} \left(1 - e^{iux} - \frac{iux}{1+x^2}\right) \lambda(dx) \right\}$$

$\lambda(dx)$: Lévy measure

<u>Lévy process</u>

Q: Does SBM exist?

Def Let $\{X_t : t \in T\} = X$ be stoch process, $\mathcal{X}$-valued, $(\mathcal{X}, \Sigma)$ = meas. space.

For $n \geq 1$, $t_1 \ldots t_n \in T$:

$$\mu_{(t_1 \ldots t_n)}(A) := P\big((X_{t_1} \ldots X_{t_n}) \in A\big)$$

$$A \in \Sigma^{\otimes n}$$

We call $\{\mu_{(t_1 \ldots t_n)} : n \geq 1, t_1 \ldots t_n \in T\}$ the finite dimensional distributions of X (FDD)

$$\Sigma \otimes \Sigma = \sigma\big(\{A_1 \times A_2 : A_1, A_2 \in \Sigma\}\big)$$

$$\Sigma^{\otimes n} = \sigma\Big(\Big\{\mathop{\times}_{i=1}^{n} A_i : A_1 \ldots A_n \in \Sigma\Big\}\Big)$$

σ-alg on $\mathcal{X}^n$

Restrictions:

- permutation invariance: $\forall n \geq 1\ \forall \sigma \in \mathcal{S}_n$

$$\mu_{(t_{\sigma(1)}, \ldots t_{\sigma(n)})}(A_{\sigma(1)} \times \ldots \times A_{\sigma(n)}) = \mu_{(t_1 \ldots t_n)}(A_1 \times \ldots \times A_n) \quad \forall t_i\ \forall A_i$$

- restriction:

$$\mu_{(t_1 \ldots t_n)}(A_1 \times \ldots \times A_{n-1} \times \mathcal{X}) = \mu_{(t_1 \ldots t_{n-1})}(A_1 \times \ldots \times A_{n-1})$$