MATH 275D take-home final exam
due at or before 11:59PM on 12/20/2025

Instructions: Solve the following problems. You may use results from class, provided you provide a full and correct statement thereof. The exam is open book but you must work alone and not discuss the exam with other students. Return in PDF format, typeset in LateX, by above due date via email to biskup@math.ucla.edu.

Problem 1: Let M be a continuous local martingale such that

$$\langle M\rangle_\infty := \lim_{t\to\infty}\langle M\rangle_t \in L^1$$

Prove that for any finite stopping time T we have

$$M_T \in L^2 \ \wedge\ E\big(M_T^2\big) = E\big(\langle M\rangle_T\big)$$

Problem 2: Let $f \in C(\mathbb{R})$ be such that, for some $g, h \in C(\mathbb{R})$ and with $\{B_t\colon t \geqslant 0\}$ denoting the standard Brownian motion started from zero,

$$f(B_t) = f(B_0) + \int_0^t g(B_s)\,\mathrm{d}B_s + \frac{1}{2}\int_0^t h(B_s)\,\mathrm{d}s$$

holds a.s. for each $t \geqslant 0$. Do the following:

(1) prove that $f \in C^2(\mathbb{R})$, and
(2) conclude that $g = f'$ and $h = f''$.

Problem 3: Let B denote Brownian motion and consider a jointly-measurable process Y adapted to the Brownian filtration. Let M be a continuous process defined via localization so that $M_t = \int_0^t Y_s\,\mathrm{d}B_s$ a.s. for all $t < \tau_\infty := \lim_{n\to\infty}\tau_n$, where

$$\tau_n := \inf\Big\{t \geqslant 0\colon \int_0^t Y_s^2\,\mathrm{d}s \geqslant n\Big\}$$

Assuming that $\int_0^\infty Y_s^2\,\mathrm{d}s = \infty$ a.s., prove that

$$\limsup_{t\uparrow\tau_\infty} M_t = +\infty \ \wedge\ \liminf_{t\uparrow\tau_\infty} M_t = -\infty$$

a.s. on $\{\tau_\infty < \infty\}$. In particular, $t \mapsto M_t$ cannot be extended continuously to $t = \tau_\infty$.

Problem 4: Let X be a standard Brownian motion started from $X_0 = x$ under the law P^x. Denote $\tau_a := \inf\{t \geqslant 0\colon X_t = a\}$ and consider the conditional laws

$$Q^x(A) := P^x(A|\tau_1 < \tau_{-1})$$

for $x \in (-1, 1)$. Prove that under Q^x, the process X solves the SDE

$$\mathrm{d}X_t = \frac{1}{1+X_t}\,\mathrm{d}t + \mathrm{d}B_t$$

with initial value $X_0 = x$ up to the first hitting time of level 1.