

29. BROWNIAN LOCAL TIME

In this final lecture, we discuss a bit the theory of the Brownian local time with interesting applications to generalized Itô formula and reflected Brownian motion.

29.1 Occupation time measure and Trotter's theorem.

An interesting as well as useful class of functionals associated with a stochastic process X arises from integrals of the form

$$\int_0^t f(X_s) ds \tag{29.1}$$

This is an object that is additive in time and, as a function of f , constitutes a linear functional that is even continuous in, e.g., the supremum norm. The functional is naturally represented by the *occupation-time measure* \mathcal{L}_t^X given for any measurable set A in the space where X takes its value as

$$\mathcal{L}_t^X(A) := \int_0^t 1_A(X_s) ds \tag{29.2}$$

Indeed, thanks to the monotone class theorem we have

$$\int_0^t f(X_s) ds = \int f(x) \mathcal{L}_t^X(dx) \tag{29.3}$$

whenever at least one of the integrals exists.

It is clear that \mathcal{L}_t^X is concentrated on the trace $\{X_s : s \in [0, t]\}$ of the process X up to time t . For this reason, with X taking values in \mathbb{R}^d we do not expect the measure \mathcal{L}_t^X to be absolutely continuous with respect to the Lebesgue measure on \mathbb{R}^d unless $d = 1$. The latter is exactly the situation we want to explore here.

Let B denote a standard Brownian motion started at a generic value of B_0 . Recall that the Tanaka formula (Lemma 24.3) tells us that

$$\begin{aligned} \frac{1}{2\epsilon} \mathcal{L}_t^B((x - \epsilon, x + \epsilon)) &= \frac{1}{2\epsilon} \int_0^t 1_{\{|B_s - x| < \epsilon\}} ds \\ &\xrightarrow[\epsilon \downarrow]{P} |B_t - x| - |B_0 - x| - \int_0^t \text{sgn}(B_s - x) dB_s \end{aligned} \tag{29.4}$$

This seems to indicate that \mathcal{L}_t^B actually has a probability density given as

$$|B_t - x| - |B_0 - x| - \int_0^t \text{sgn}(B_s - x) dB_s \tag{29.5}$$

at $x \in \mathbb{R}$. Unfortunately, the above limit statement is not strong enough to make this into a proof because, while we can upgrade the convergence to a.s. by making ϵ approach zero along a summable sequence, the implicit null set may depend on x which prevents us from treating more than countably many x at the same time. We will therefore proceed differently by showing that the object on the right does admit a version which is continuous in both x and t and then check that the resulting quantity serves as a density of the occupation-time measure \mathcal{L}_t^B .

Before we do this, let us use the identities

$$|z| = 2z_+ - z \quad \text{and} \quad \text{sgn}(z - x) = 1 - 21_{(-\infty, x]}(z) \quad (29.6)$$

along with

$$(B_t - x) - (B_0 - x) = \int_0^t 1 \, dB_s \quad (29.7)$$

to rewrite (29.5) as

$$2 \left[(B_t - x)_+ - (B_0 - x)_+ + \int_0^t 1_{(-\infty, x]}(B_s) dB_s \right] \quad (29.8)$$

We will actually work with this expression instead of (29.5) as it is technically somewhat easier. With this agree we now make a statement which, although the local time theory was developed by P. Lévy already in the 1930s, found a correct proof only in the 1950s:

Theorem 29.1 (Trotter 1958) *Consider a probability space $(\Omega, \mathcal{F}, P^0)$ supporting a standard Brownian motion B with P^0 such that $P^0(B_0 = 0) = 1$. There exists a continuous stochastic process $\{I(t, x) : t \geq 0, x \in \mathbb{R}\}$ for which*

$$\forall t \geq 0 \forall x \in \mathbb{R}: I(t, x) = \int_0^t 1_{(-\infty, x]}(B_s) dB_s \quad P^0\text{-a.s.} \quad (29.9)$$

and, setting

$$L_t(x) := 2 \left[(B_t - x)_+ - (B_0 - x)_+ + I(t, x) \right] \quad (29.10)$$

which is continuous in both t and x , there exists $\Omega^* \in \mathcal{F}$ with $P^0(\Omega^*) = 1$ such that

$$\forall t \geq 0 \forall A \in \mathcal{B}(\mathbb{R}): \mathcal{L}_t^B(A) = \int_A L_t(x) dx \quad \text{on } \Omega^* \quad (29.11)$$

In particular, on Ω^* , for every $t \geq 0$ the occupation-time measure \mathcal{L}_t^B is absolutely continuous with respect to the Lebesgue measure on \mathbb{R} with L_t serving as the Radon-Nikodym derivative.

We will refer to $L_t(x)$ as the *Brownian local time* at x . As we will see in the proof, the map $t, x \mapsto L_t(x)$ is actually (locally) γ -Hölder continuous for every $\gamma < 1/2$.

29.2 Proof of Trotter's theorem.

The proof will require the following lemma (generalizing the inequality (20.37) from the proof of Lemma 20.5):

Lemma 29.2 (A Burkholder-Davis-Gundy type inequality) *For all natural $n \geq 1$ there exists $A_n \in (0, \infty)$ such that for all $M \in \mathcal{M}_{\text{loc}}^{\text{cont}}$ with $M_0 = 0$ and all $t \geq 0$,*

$$E(M_t^{2n}) \leq A_n E(\langle M \rangle_t^n) \quad (29.12)$$

We leave the proof to a reference to basic textbooks. Instead we focus on:

Proof of Theorem 29.1. The idea of the proof is simple: Use estimates to control the moments of the difference of the stochastic integral in (29.9) at two different t 's and x 's. Then feed this into the criterion in Kolmogorov-Čenstov theorem to get existence of a continuous version. Finally, employ computations to check that this indeed obeys (29.11).

Pick $0 \leq u < t$ and $x < y$. Then the additivity of stochastic integrals along with the inequality $(a + b)^{2n} \leq 4^n(a^{2n} + b^{2n})$ and Lemma 29.2 yields

$$\begin{aligned}
 & E \left(\left| \int_0^t 1_{(-\infty, y]}(B_s) dB_s - \int_0^u 1_{(-\infty, x]}(B_s) dB_s \right|^{2n} \right) \\
 &= E \left(\left| \int_0^t 1_{(u, t]}(s) 1_{(-\infty, x]}(B_s) dB_s + \int_0^u 1_{(-x, y]}(B_s) dB_s \right|^{2n} \right) \\
 &\leq 4^n E \left(\left| \int_0^t 1_{(u, t]}(s) 1_{(-\infty, x]}(B_s) dB_s \right|^{2n} \right) + 4^n E \left(\left| \int_0^u 1_{(-x, y]}(B_s) dB_s \right|^{2n} \right) \\
 &\leq 4^n A_n E \left(\left| \int_u^t 1_{(-\infty, x]}(B_s) ds \right|^n \right) + 4^n A_n E \left(\left| \int_0^u 1_{(-x, y]}(B_s) ds \right|^n \right)
 \end{aligned} \tag{29.13}$$

(The above used that $u \leq t$ but the same estimate is obtained when $u > t$ as well.) The first expectation on the right is bounded by $|t - u|^n$. For the second expectation, we write the n -th power of the integral as a product of integrals and then use the reordering trick to write it as an integral over increasing sequences of times. This yields

$$\begin{aligned}
 & E \left(\left| \int_0^u 1_{(-x, y]}(B_s) ds \right|^n \right) \\
 &= n! \int_{0 \leq s_1 < \dots < s_n \leq t} ds_1 \dots ds_n \int_{\mathbb{R}^n} dx_1 \dots dx_n \prod_{i=1}^n (f_{s_i - s_{i-1}}(x_i - x_{i-1}) 1_{(x, y]}(x_i))
 \end{aligned} \tag{29.14}$$

where f_s is the probability density of $\mathcal{N}(0, s)$ and $s_0 := 0$ and $x_0 := 0$. Using that $f_s \leq (2\pi s)^{-1/2}$ we then get

$$E \left(\left| \int_0^u 1_{(-x, y]}(B_s) ds \right|^n \right) \leq n! |x - y|^n \left(\int_0^t \frac{1}{\sqrt{2\pi s}} \right)^n \tag{29.15}$$

Combining these estimates we conclude that, for all $(t, y), (u, x) \in \mathbb{R}_+ \times \mathbb{R}$,

$$E \left(\left| \int_0^t 1_{(-\infty, y]}(B_s) dB_s - \int_0^u 1_{(-\infty, x]}(B_s) dB_s \right|^{2n} \right) \leq C_n (|t - s| + |x - y|)^n \tag{29.16}$$

with $C_n := 4^n A_n [1 + n!(2t/\pi)^{n/2}]$. The Kolmogorov-Čenstov Theorem in \mathbb{R}^2 then implies existence of a γ -Hölder continuous version for each $\gamma := \frac{n-1}{2n}$. As this tends to $1/2$ with $n \rightarrow \infty$, we have γ -Hölder continuity for all $\gamma < 1/2$.

Call the continuous version $I(t, x)$. In order to prove the second part of the claim, let $f \in C_c^2(\mathbb{R})$ and note that the Itô formula gives

$$\begin{aligned}
 \int f''(x) \mathcal{L}_t^B(dx) &= \int_0^t f''(B_s) ds \\
 &= 2 \left[f(B_t) - f(B_0) - \int_0^t f'(B_s) dB_s \right]
 \end{aligned} \tag{29.17}$$

For the stochastic integral we now invoke the Fundamental Theorem of Calculus along with the fact that f' vanishes for large positive arguments to get

$$\int_0^t f'(B_s) dB_s = - \int_0^t \left(\int_{\mathbb{R}} f''(y) 1_{(-\infty, y]}(B_s) dy \right) dB_s \quad (29.18)$$

where we also noted that $1_{(-\infty, y]}(B_s) = 1_{[B_s, \infty)}(y)$. We now invoke a variant of “Fubini for Itô” lemmas whose proof we to homework:

Lemma 29.3 For all $h \in L^\infty(\mathbb{R})$ with compact support

$$\int_0^t \left(\int_{\mathbb{R}} h(y) 1_{(-\infty, y]}(B_s) dy \right) dB_s = \int_{\mathbb{R}} h(y) \left(\int_0^t 1_{(-\infty, y]}(B_s) dB_s \right) dy \quad P^0\text{-a.s.} \quad (29.19)$$

where the inner integral on the right should be interpreted as $I(t, y)$.

Using also the definition of $L_t(y)$, this yields

$$\begin{aligned} - \int_0^t f'(B_s) dB_s &= \frac{1}{2} \int_{\mathbb{R}} f''(y) L_t(y) dy \\ &\quad - \int_{\mathbb{R}} f''(y) (B_t - y)_+ dy + \int_{\mathbb{R}} f''(y) (B_0 - y)_+ dy \end{aligned} \quad (29.20)$$

Integration by parts (using also that f' has compact support) now shows that

$$\int_{\mathbb{R}} f''(y) (z - y)_+ dy = \int_{-\infty}^z f'(y) dy = f(z) \quad (29.21)$$

and so

$$f(B_t) - f(B_0) - \int_0^t f'(B_s) dB_s = \frac{1}{2} \int_{\mathbb{R}} f''(y) L_t(y) dy \quad P^0\text{-a.s.} \quad (29.22)$$

Plugging this in (29.17) while relabelling f'' for f shows

$$\forall f \in C_c(\mathbb{R}): \quad \int f''(x) \mathcal{L}_t^B(dx) = \int f(y) L_t(y) dy \quad P^0\text{-a.s.} \quad (29.23)$$

Taking $f \uparrow 1_{(a,b)}$ along a sequence in turn gives

$$\forall a < b: \quad \mathcal{L}_t^B((a, b)) = \int_a^b L_t(x) dx \quad P^0\text{-a.s.} \quad (29.24)$$

by way of the Monotone Convergence Theorem.

The implicit null set in (29.24) may depend on a, b and t , so let $\Omega_{a,b,t}$ be the set where the equality holds there. Then set

$$\Omega^* := \bigcap_{\substack{a,b,t \in \mathbb{Q} \\ a < b, t \geq 0}} \Omega_{a,b,t} \quad (29.25)$$

Then $\Omega^* \in \mathcal{F}$ and $P^0(\Omega^*) = 1$. On Ω^* the equality holds for all bounded open intervals with rational endpoints. Since these intervals form a π -system that generates all Borel sets, Dynkin’s π/λ -theorem gives (29.11) simultaneously for all rational $t \geq 0$. But $|L_{t'}^B(A) - L_t^B(A)| \leq |t - t'|$ so both sides of the equality (29.11) are continuous in t . Hence equality holds for all $t \geq 0$, rational or not. \square

As a consequence we get:

Corollary 29.4 *Any jointly continuous version of Brownian local time $\{L_t(x) : t \geq 0, x \in \mathbb{R}\}$ is such that $t \mapsto L_t(x)$ is non-decreasing at every $x \in \mathbb{R}$ on Ω^* such that (29.11) holds.*

Proof. Assume containment in Ω^* and, given $t < t'$, let $A := \{x \in \mathbb{R} : L_t(x) > L_{t'}(x)\}$. Then $\mathcal{L}_t^B(A) \leq \mathcal{L}_{t'}^B(A)$ via (29.11) implies that A is a Lebesgue-null set. But continuity of $x \mapsto L_t(x)$ forces A to be open and so A must be empty. \square

We remark that there is another parameter with respect to which we may want to have continuity: the starting point of the Brownian motion. This is achieved by interpreting the problem on the Wiener space and defining $I(t, x)$ for general B_0 by way of a shift of the whole path.

29.3 Applications.

The concept of the Brownian local time is very useful in treating lots of singular scenarios. One of these is the *reflected Brownian motion*. For B the standard Brownian motion started from $B_0 > -a$, we interpret this as the process

$$X_t := -a + |B_t + a| \tag{29.26}$$

This process runs as Brownian motion until it hits level $-a$. Then all excursions below $-a$ are “flipped” above $-a$, while those above are kept. Tanaka’s formula gives us

$$X_t = |B_t + a| - |B_0 + a| = \int_0^t \text{sgn}(B_s + a) dB_s + L_t(-a) \tag{29.27}$$

which gives the infinitesimal version of the form

$$dX_t = \text{sgn}(B_t + a) dB_t + dL_t(-a) \tag{29.28}$$

The first term is still a Brownian motion while the interpretation of the second term is a process that increases only when X hits $-a$ at which point it receives a “kick” to stay above $-a$, should it try to go below instead.

Another application is the following generalization of Itô’s formula:

Theorem 29.5 (Generalized Itô formula) *Let B be a standard Brownian motion and let $f : \mathbb{R} \rightarrow \mathbb{R}$ be continuous such that*

- (1) *the right derivative*

$$D^+ f(x) := \lim_{h \downarrow 0} \frac{f(x+h) - f(x)}{h} \tag{29.29}$$

exists at all $x \in \mathbb{R}$, and

- (2) *$D^+ f$ is right continuous and bounded variation on compact intervals.*

Let μ be the unique signed Radon measure such that

$$\forall a < b: \quad \mu((a, b]) = D^+ f(b) - D^+ f(a) \tag{29.30}$$

Then

$$f(B_t) = f(B_0) + \int_0^t D^+ f(B_s) dB_s + \frac{1}{2} \int_{\mathbb{R}} L_t(x) \mu(dx) \tag{29.31}$$

holds a.s. for all $t \geq 0$.

The proof of this is a direct application of the ordinary Itô formula to a smoothed-over version of f and taking suitable limits. We leave the details to the reader. We note that, as a corollary, we settled a debt by proving:

Proof of Lemma 24.4. If $f \in C^1(\mathbb{R})$ with f' absolutely continuous, the Lebesgue differentiation theorem ensures existence of f'' such that

$$f'(b) - f'(a) = \int_a^b f''(x) dx \quad (29.32)$$

But this means that μ above takes the form $\mu(dx) = f''(x)dx$ and so

$$\int_{\mathbb{R}} L_t(x) \mu(dx) = \int_{\mathbb{R}} L_t(x) f''(x) dx = \int_0^t f''(B_s) ds \quad (29.33)$$

by way of the identity (29.3). (Since L_t is continuous and thus bounded, the integral exists whenever f'' is locally integrable.) \square

Further reading: Karatzas-Shreve, Section 3.6B