28. Novikov's condition

Here we elaborate further on the condition that makes Girsanov's theorem work which also reveals its full range of applicability.

28.1 Subtlety of the condition $EM_t = 1$.

We start by noting that the requirement that the local martingale M_t has unit expectation at a given $t \geqslant 0$ is not at all trivial. Indeed, as shown in the proof of Lemma 27.5, the condition $EM_t = 1$ guarantees that, under the exponential tilting, no mass escapes to "infinity" when we take the stopping times τ_n truncating the exponent to infinity. Not surprisingly, the condition is also equivalent to $\widetilde{P}$ in (27.18) being non-defective. We demonstrate these subtleties on the following example:

Lemma 28.1 *Let $d > 2$ and let X_t be d-dimensional Bessel process solving the SDE*

$$\mathrm{d}X_t = \frac{d-1}{2X_t}\mathrm{d}t + \mathrm{d}B_t \tag{28.1}$$

with initial value $X_0 = 1$. Then $M_t := X_t^{2-d}$ takes the form

$$M_t = \exp\left\{-(d-2)\int_0^t \frac{1}{X_s}\mathrm{d}B_s - \frac{(d-2)^2}{2}\int_0^t \frac{1}{X_s^2}\,\mathrm{d}s\right\} \tag{28.2}$$

which is well defined for all $t \geqslant 0$ as $\inf_{t\geqslant 0} X_t > 0$ a.s. and is a local martingale. Yet

$$\forall t > 0\colon \quad EM_t < 1 \tag{28.3}$$

Proof. That M_t takes the form (28.2) is checked via

$$\begin{aligned} M_t = X_t^{2-d} &= \exp\bigl\{(2-d)\log X_t\bigr\} \\ &= \exp\left\{(2-d)\int_0^t \frac{1}{X_s}\mathrm{d}X_s - \frac{2-d}{2}\int_0^t \frac{1}{X_s^2}\mathrm{d}\langle X\rangle_s\right\} \end{aligned} \tag{28.4}$$

Plugging (28.1) for $\mathrm{d}X_s$ we then get (28.1).

Suppose, by way of contradiction, that $EM_T = 1$ for some $T > 0$. Then Theorem 27.4 applies and gives that $\{\widetilde{B}_t\colon t \leqslant T\}$, for

$$\widetilde{B}_t := B_t + \int_0^t \frac{d-2}{X_s}\mathrm{d}s \tag{28.5}$$

is a standard Brownian motion. As this gives $\mathrm{d}B_t = \mathrm{d}\widetilde{B}_t - \frac{d-2}{X_t}\mathrm{t}$, in terms of $\widetilde{B}$ the SDE (28.1) transforms into

$$\mathrm{d}X_t = \Bigl(\frac{d-1}{2X_t} - \frac{d-2}{X_t}\Bigr)\mathrm{d}t + \mathrm{d}\widetilde{B}_t = \frac{3-d}{2X_t}\mathrm{d}t + \mathrm{d}\widetilde{B}_t \tag{28.6}$$

Setting $d' - 1 := 3 - d$ gives $d' = 4 - d$ and so we get that $\{X_t\colon t \leqslant T\}$ is a $(4-d)$-dimensional Bessel process under $\widetilde{P}$. (This uses that the conditions for strong uniqueness hold for Bessel's SDE.)

The key point for the proof is that, since $4 - d < 2$ for $d > 2$, the $(4-d)$-dimensional Bessel process hits zero in time interval $[0, T]$ with positive probability. (In Theorem 21.5

we proved that this happens for T sufficiently large but one can adjust the argument to make it work for all $T > 0$.) This means

$$\widetilde{P}\Big(\inf_{t\in[0,T]} X_t = 0\Big) > 0 \tag{28.7}$$

yet the assumption that $d > 2$ gives (see Theorem 21.5(1))

$$P\Big(\inf_{t\in[0,T]} X_t = 0\Big) = 0 \tag{28.8}$$

in contradiction with $\widetilde{P} \ll P$ on $\mathcal{F}_T$. It follows that $EM_T < 1$ after all. □

The proof shows the mechanics by which $EM_t = 1$ fails: The exponential change of measure pushes some part of the mass of P outside the support of P. An interesting additional twist to the above example is that the local martingale $\{M_t \colon t \geqslant 0\}$ in (28.2) is actually uniformly integrable which provides an example of a UI local martingale which is not a martingale. We check this for integer $d \geqslant 3$:

Lemma 28.2 *Let $d \in \mathbb{N}$ obey $d \geqslant 3$ and let M_t be as in* (28.2). *Then*

$$\forall p \in \big[0, \tfrac{d}{d-2}\big): \quad \sup_{t\geqslant 0} E(M_t^p) < \infty \tag{28.9}$$

In particular, M is uniformly integrable.

Proof. Let B be the d-dimensional standard Brownian motion started from zero. Given a point $x \in \mathbb{R}^d$ with $|x| = 1$, we then have $M_t \overset{\text{law}}{=} |x + B_t|^{2-d}$ and so we have

$$P\big(|M_t| > \lambda\big) = P\big(|x + B_t| < \lambda^{-\frac{1}{d-2}}\big) \tag{28.10}$$

On the Euclidean ball of radius $1/2$ centered at x, the probability density of B_t is bounded by $(2\pi t)^{-d/2}\mathrm{e}^{-\frac{1}{8t}}$. For $\lambda > 2^{d-2}$ we thus have

$$P\big(|M_t| > \lambda\big) \leqslant \pi_d \frac{\mathrm{e}^{-\frac{1}{8t}}}{(2\pi t)^{d/2}} \lambda^{-\frac{d}{d-2}} \tag{28.11}$$

where π_d is the volume of Euclidean ball of radius 1 in $\mathbb{R}^d$. A distribution with these kinds of tails has moments up to, but possibly not including, $\frac{d}{d-2}$. Since the prefactor is bounded uniformly in $t \geqslant 0$, we get the claim. □

We note that, since the probability density of the d-dimensional Bessel is known explicitly, a similar calculation shows that the same holds for all real $d > 2$.

28.2 Conditions for M to be a martingale.

In light of above observations, one is thus interested in natural, and reasonably sharp, conditions on the process $Y \in \mathcal{V}^{\text{loc}}$ in

$$M_t := \exp\Big\{\int_0^t Y_s \mathrm{d}B_s - \frac{1}{2}\int_0^t Y_s^2 \mathrm{d}s\Big\} \tag{28.12}$$

to guarantee $EM_t = 1$. We will work in more generality by doing the same for

$$M_t := \exp\{X_t - \tfrac12\langle X\rangle_t\} \tag{28.13}$$

with a general $X \in \mathscr{M}^{\text{cont}}_{\text{loc}}$. Here is one that is fairly easy to prove:

Lemma 28.3 *Suppose that M is given by* (28.13) *with some $X \in \mathscr{M}^{\text{cont}}_{\text{loc}}$. Then $M \in \mathscr{M}^{\text{cont}}_{\text{loc}}$. Moreover, given $t > 0$, if*

$$\exists\epsilon > 0\colon\quad E\biggl(\exp\Bigl\{\Bigl(\frac12 + \epsilon\Bigr)\langle X\rangle_t\Bigr\}\biggr) < \infty \tag{28.14}$$

then $EM_t = 1$ and $\{M_{s\wedge t}\colon s \geqslant 0\} \in \mathscr{M}^{\text{cont}}$.

Proof. Set $\tau_n := \inf\{t \geqslant 0\colon |X_t| \geqslant n\}$. As is easy to check using the Itô formula, $\{M_{t\wedge\tau_n}\colon t \geqslant 0\}$ is a bounded local martingale and thus a martingale. Pick $\lambda > 1$ and $p > 1$ and let q be such that $p^{-1} + q^{-1} = 1$. The Hölder inequality then gives

$$\begin{aligned} E\bigl(M^\lambda_{\tau_n\wedge t}\bigr) &= E\bigl(\mathrm{e}^{\lambda X_{\tau_n\wedge t} - \frac{\lambda}{2}\langle X\rangle_{\tau_n\wedge t}}\bigr) \\ &= E\bigl(\mathrm{e}^{\lambda X_{\tau_n\wedge t} - \frac{p\lambda^2}{2}\langle X\rangle_{\tau_n\wedge t}}\mathrm{e}^{\frac{\lambda}{2}(p\lambda-1)\langle X\rangle_{\tau_n\wedge t}}\bigr) \\ &\leqslant \Bigl(E\bigl(\mathrm{e}^{p\lambda X_{\tau_n\wedge t} - \frac{p^2\lambda^2}{2}\langle X\rangle_{\tau_n\wedge t}}\bigr)\Bigr)^{1/p}\Bigl(E\bigl(\mathrm{e}^{\frac{\lambda}{2}(p\lambda-1)q\langle X\rangle_{\tau_n\wedge t}}\bigr)\Bigr)^{1/q} \end{aligned} \tag{28.15}$$

Since, similarly as argued above, $\{\mathrm{e}^{p\lambda X_{\tau_n\wedge t} - \frac{p^2\lambda^2}{2}\langle X\rangle_{\tau_n\wedge t}}\colon t \geqslant 0\}$ is a bounded local martingale, and thus a martingale, the first expectation on the right equals one. The monotonicity of $t \mapsto \langle X\rangle_t$ along with the Monotone Convergence Theorem then gives

$$\sup_{n\geqslant1} E\bigl(M^\lambda_{\tau_n\wedge t}\bigr) \leqslant \Bigl(E\bigl(\mathrm{e}^{\frac{\lambda}{2}(p\lambda-1)q\langle X\rangle_t}\bigr)\Bigr)^{1/q} \tag{28.16}$$

Noting that

$$\lambda(p\lambda - 1)q = p\lambda\frac{p\lambda - 1}{p - 1} \tag{28.17}$$

tends to one as $\lambda \downarrow 1$ and $p \downarrow 1$, we can choose $\lambda > 1$ and $p > 1$ so that the coefficient multiplying $\langle X\rangle_t$ in (28.16) is less than $\frac12 + \epsilon$. The condition (28.14) then gives that $\{M_{\tau_n\wedge t}\colon n \geqslant 1\}$ is bounded uniformly in L^λ and is thus uniformly integrable. In light of $E(M_{\tau_n\wedge t}) = 1$ and $M_{\tau_n\wedge t} \to M_t$ a.s., this yields $EM_t = 1$. □

Note that Lemma 28.3 applies in the context (28.12) whenever Y is locally bounded. As it turns out, the integrability condition (28.14) is actually not far from optimal. First, as examples show, the conclusion does not hold in general if (28.14) with $\epsilon < 0$ is assumed. Moreover, the case $\epsilon = 0$ is included in the positive conclusion, albeit with a different (and somewhat complicated) proof:

Theorem 28.4 (Novikov's condition) *Suppose $M \in \mathscr{M}^{\text{cont}}_{\text{loc}}$ is derived from $X \in \mathscr{M}^{\text{cont}}_{\text{loc}}$ as in* (28.13). *Then for all $t \geqslant 0$,*

$$E\Bigl(\exp\{\tfrac12\langle X\rangle_t\}\Bigr) < \infty \tag{28.18}$$

implies $EM_t = 1$.

As it turns out, (28.18) can further be weakened by requiring the expectation to be finite with t replaced by t', for any $t' < t$. This is known as the *Kazamaki condition*. The author of these notes believes that, while sufficient, the Kazamaki condition (and thus also Novikov's condition) is not necessary. For the proof of Theorem 28.4 we refer the reader to the literature.

The main point why Novikov's condition is so well known is that it is easy to state (various sufficient conditions exist for Y of the form $Y_s = a(B_s)$ that imply it) and does not involve extraneous parameters. Looking for sharper conditions may not be very practical because the closer to optimal, the harder the condition will likely be to verify.

28.3 Applications of Girsanov's theorem.

We now move to applications of Girsanov's theorem. The first one is on solving a particular class of SDEs. The following expands on Corollary 27.6:

Theorem 28.5 (Removal of drift term) *Suppose $a\colon [0,\infty)\times\mathbb{R}\to\mathbb{R}$ is Borel measurable and such that, for B a standard Brownian motion started from zero and $x\in\mathbb{R}$,*

$$\forall t\geqslant 0\colon\quad E\left(\exp\left\{\frac12\int_0^t a(s,x+B_s)^2\mathrm{d}s\right\}\right)<\infty \tag{28.19}$$

Then the SDE

$$\mathrm{d}X_t = a(t,X_t)\mathrm{d}t+\mathrm{d}B_t \tag{28.20}$$

admits a (global) weak solution with $X_0 = x$.

Proof. Let $(\Omega,\mathcal{F},P)$ be a probability space supporting a Brownian motion B and the Brownian filtration $\{\mathcal{F}_t^B\}_{t\geqslant 0}$, where $\mathcal{F}_t^B := \sigma(B_s\colon s\leqslant t)$. For each $t\geqslant 0$, let $\widetilde{P}_t$ be the measure on$(\Omega,\mathcal{F}_t)$ defined by $\widetilde{P}_t(A) := E(1_A M_t)$, where E is expectation with respect to P and M_t is as in (27.35). Thanks to Theorem 28.4, (28.19) guarantees that $\widetilde{P}_t$ is a probability measure for all $t\geqslant 0$. The fact that M is a martingale also gives that

$$\forall s\leqslant t\,\forall A\in\mathcal{F}_s\colon\quad \widetilde{P}_t(A)=\widetilde{P}_s(A) \tag{28.21}$$

which shows that $\widetilde{P}_t$ is a restriction of an additive set function on $\bigcup_{t\geqslant 0}\mathcal{F}_t^B$. The map $\omega\mapsto B(\omega)$ identifies the latter with an algebra of sets generating the product σ-algebra in $\mathbb{R}^{[0,\infty)}$. Denoting

$$\mathcal{F}_\infty^B := \sigma\left(\bigcup_{t\geqslant 0}\mathcal{F}_t^B\right) \tag{28.22}$$

Kolmogorov Extension Theorem then extends the above additive set function to a unique probability measure $\widetilde{P}$ on $(\Omega,\mathcal{F}_\infty^B)$.

The argument in the proof of Corollary 27.6 now ensures that, under $\widetilde{P}$, the process X solves the SDE up to time t, for all $t\geqslant 0$. Hence it is a global weak solution. □

The caution we exercised in constructing $\widetilde{P}$ on only the terminal point $\mathcal{F}_\infty^B$ of the minimal Brownian filtration and not a larger σ-algebra — for instance, its augmented version $\widetilde{\mathcal{F}}_\infty^B := \sigma(\mathcal{N}\cup\bigcup_{t\geqslant 0}\mathcal{F}_t^B)$ where $\mathcal{N}$ are the null sets of P — is necessitated by the fact that $\widetilde{P}$ may *not* be absolutely continuous with respect to P on $(\Omega, \mathcal{F}^B_\infty)$, in spite of it being so on $(\Omega, \mathcal{F}^B_t)$ for every $t \in [0,\infty)$. This will be demonstrated in our next application on the Brownian motion conditioned to stay positive.

Since the (one-dimensional) standard Brownian motion is recurrent, we cannot condition on $\inf_{t\geqslant 0} B_t > 0$ as the latter is a null set no matter how large positive B_0 is chosen to be. In order to define the notion of "Brownian motion conditioned to stay positive" we will thus need to employ a limit procedure.

Let B be the standard Brownian motion started from $x > 0$ defined on a probability space $(\Omega, \mathcal{F}, P)$. As usual, denote $\tau_a := \inf\{t \geqslant 0 \colon B_t = a\}$ and, for each $T > 0$, consider the measure Q^x_t on $(\Omega, \mathcal{F})$ defined by

$$Q^x_T(A) := P(A|\tau_0 > T) \tag{28.23}$$

Note that here we stipulate only $\inf_{t\in[0,T]} B_t > 0$ which has positive probability as soon as B_0 is positive. We then have:

Theorem 28.6 (Brownian motion conditioned to stay positive) *For all $x > 0$, all $t \geqslant 0$ and all $A \in \mathcal{F}_t$, the limit*

$$Q^x(A) := \lim_{T\to\infty} Q^x_T(A) \tag{28.24}$$

exists and extends to a unique probability measure on $(\Omega, \mathcal{F}^B_\infty)$, where $\mathcal{F}^B_\infty$ is as above. Moreover, the process $\{B_t \colon t \geqslant 0\}$ under Q^x has the law of the 3-dimensional Bessel process started from x.

Proof. Write P^x for the law of the standard Brownian motion started from x and let E^x be the associated expectation. Define

$$h_t(x) := P^x(\tau_0 > t) \tag{28.25}$$

Then for each $0 \leqslant t < T$ and $A \in \mathcal{F}_t$, the Markov property gives

$$Q^x_T(A) = E^x\biggl(1_{A\cap\{\tau_0>t\}}\frac{h_{T-t}(B_t)}{h_T(x)}\biggr) \tag{28.26}$$

We now claim:

Lemma 28.7 (Reflection principle) *For each $t \geqslant 0$ and $x > 0$,*

$$h_t(x) = P^x(B_t > 0) - P^x(B_t < 0) \tag{28.27}$$

In particular, we have

$$\forall x \geqslant 0\,\forall t > 0\colon\quad 2\mathrm{e}^{-\frac{x^2}{2t}}\sqrt{\frac{2}{\pi}}\,\frac{x}{\sqrt{t}} \leqslant h_t(x) \leqslant \sqrt{\frac{2}{\pi}}\,\frac{x}{\sqrt{t}} + \frac{x^2}{t} \tag{28.28}$$

and

$$\forall x \geqslant 0\colon\quad \lim_{t\to\infty}\sqrt{t}\,h_t(x) = \sqrt{\frac{2}{\pi}}\,x \tag{28.29}$$

Postponing the proof of this lemma until the main line of argument is finished, plugging this into the ratio on the right of (28.26) we now observe that, for $T \geqslant 2t$, $T \geqslant 2$ and

and $x \leqslant \sqrt{T}$,

$$\frac{h_{T-t}(B_t)}{h_T(x)} \leqslant \frac{\sqrt{\mathrm{e}}}{x}\frac{\sqrt{T}}{\sqrt{T-t}}\Bigl(B_t + 2\frac{B_t^2}{\sqrt{T-t}}\Bigr) \leqslant \frac{\sqrt{\mathrm{e}}}{x}(B_t + 2B_t^2) \tag{28.30}$$

where we relied on (28.28) along with $\sqrt{\pi/2} \leqslant 2$ and $\mathrm{e}^{-\frac{x^2}{2T}} \geqslant \mathrm{e}^{-1/2}$ when $x \leqslant \sqrt{T}$. The limit statement (28.29) in turn gives

$$\lim_{T\to\infty} \frac{h_{T-t}(B_t)}{h_T(x)} = \frac{B_t}{x} \tag{28.31}$$

The Dominated Convergence Theorem then allows us to conclude that

$$\lim_{T\to\infty} Q_T^x(A) = E^x\Bigl(1_{A\cap\{\tau_0>t\}}\frac{B_t}{B_0}\Bigr) \tag{28.32}$$

In particular, the limit in (28.24) exists. The argument used in the proof of Theorem 28.5 shows that Q^x extends to a probability measure on $(\Omega, \mathcal{F}_\infty^B)$.

In order to identify the law of B under Q^x, we need a minor truncation argument. Pick $a > 0$ and use the Itô formula to write

$$\frac{B_t}{B_0} = \exp\bigl\{\log B_t - \log B_0\bigr\} = M_t \quad \text{on } \{\tau_a > t\} \tag{28.33}$$

where

$$M_t := \exp\Bigl\{\int_0^t \frac{1}{B_s} 1_{\{\tau_a>s\}}\mathrm{d}B_s - \frac12\int_0^t \frac{1}{B_s^2}1_{\{\tau_a>s\}}\mathrm{d}s\Bigr\} \tag{28.34}$$

Let $\widetilde{P}^x$ be the extension to $\mathcal{F}_\infty^B$ of $A \mapsto E^x(1_A M_t)$ on $\mathcal{F}_t^B$. Then

$$\forall A \in \mathcal{F}_t^B\colon \quad Q^x\bigl(A \cap \{\tau_a > t\}\bigr) = \widetilde{P}^x\bigl(A\cap\{\tau_a > t\}\bigr) \tag{28.35}$$

Since $s \mapsto B_s^{-1}1_{\{\tau_a>s\}}$ is bounded, Girsanov's Theorem applies and shows that

$$\widetilde{B}_t := B_t - \int_0^t \frac{1}{B_s}1_{\{\tau_a>s\}}\mathrm{d}s \tag{28.36}$$

is a standard Brownian motion under $\widetilde{P}^x$. Turning this around, this means that B satisfies

$$\mathrm{d}B_t = \frac{1}{B_t}1_{\{\tau_a>t\}}\mathrm{d}t + \mathrm{d}\widetilde{B}_t \tag{28.37}$$

on $(\Omega, \mathcal{F}_\infty^B, P^x)$. In particular, $\{B_{\tau_a\wedge t}\colon t \geqslant 0\}$ under $\widetilde{P}^x$ is a 3-dimensional Bessel process started at x and stopped upon hitting level a.

By Theorem 21.5(1), the 3-dimensional Bessel process stays bounded from zero with probability one. Hence, taking $a \downarrow 0$ in (28.35), the finite-dimensional distributions of $\{B_t\colon t \geqslant 0\}$ under Q^x are those of the 3-dimensional Bessel process. In light of continuity of B, this proves the claim. □

Notice that Q^x is concentrated on paths that avoid zero yet P^x is not. This is actually quite common when exponential change of measure over all positive times is invoked. A condition that ensures that $\widetilde{P}$ and P remain absolutely continuous on $\mathcal{F}_\infty^B$ is that M admits a terminal element; i.e., $M_t \to M_\infty$ in L^1.

28.4 Proof of Reflection Principle.

We owe to the reader:

Proof of Lemma 28.7. The identity (28.27) is classical, proved originally for random walks in consideration of so called Ballot Theorem by J.L.F. Bertrand in 1887 (with an earlier proof from 1878 due to W.A. Whitworth). We follow a continuum version of the elegant proof of the latter by Désiré André (also from 1887). The argument relies on a reflection trick; hence the title of the lemma.

The starting point is the disjoint decomposition

$$\{\tau_0 \leqslant t\} = \Bigl(\{\tau_0 \leqslant t\} \cap \{B_t \geqslant 0\}\Bigr) \cup \Bigl(\{\tau_0 \leqslant t\} \cap \{B_t < 0\}\Bigr) \tag{28.38}$$

Next observe that, by the strong Markov property, for $x > 0$,

$$P^x\bigl(\tau_0 \leqslant t \,\wedge\, B_t > 0\bigr) = E^x\bigl(1_{\{\tau_0 \leqslant t\}} P^0(B_{t-s} > 0)|_{s:=\tau_0}\bigr) \tag{28.39}$$

Brownian symmetries imply that $P^0(B_{t-s} > 0) = P^0(B_{t-s} < 0)$ and so, wrapping this around, we conclude

$$P^x\bigl(\tau_0 \leqslant t \,\wedge\, B_t > 0\bigr) = P^x\bigl(\tau_0 \leqslant t \,\wedge\, B_t < 0\bigr) = P^x(B_t < 0) \tag{28.40}$$

where we also noticed that $B_t < 0$ implies $\tau_0 \leqslant t$ when $B_0 > 0$. Since $P^x(B_t = 0) = 0$, we thus get

$$\begin{aligned} P^x(\tau_0 > t) &= 1 - P^x(\tau_0 \leqslant t) \\ &= 1 - 2P^x(B_t < 0) = P^x(B_t > 0) - P^x(B_t < 0) \end{aligned} \tag{28.41}$$

thus proving (28.27).

In order to prove the remaining statements, we now write

$$\begin{aligned} P^x(B_t > 0) - P^x(B_t < 0) &= \int_0^\infty \frac{1}{\sqrt{2\pi t}}\bigl[\mathrm{e}^{-\frac{(y-x)^2}{2t}} - \mathrm{e}^{-\frac{(y+x)^2}{2t}}\bigr]\mathrm{d}y \\ &= \mathrm{e}^{-\frac{x^2}{2t}} \int_0^\infty \frac{1}{\sqrt{2\pi t}} \mathrm{e}^{-\frac{y^2}{2t}}\bigl[\mathrm{e}^{\frac{xy}{t}} - \mathrm{e}^{-\frac{xy}{t}}\bigr]\mathrm{d}y \end{aligned} \tag{28.42}$$

Writing the difference of the two exponentials as an integral yields

$$\begin{aligned} h_t(x) &= \sqrt{\frac{2}{\pi}}\,\frac{x}{t^{3/2}}\,\mathrm{e}^{-\frac{x^2}{2t}} \int_{-1}^1 \frac{1}{2}\Bigl(\int_0^\infty y\mathrm{e}^{-\frac{y^2}{2t} - \frac{yx}{t}s}\mathrm{d}y\Bigr)\mathrm{d}s \\ &= \sqrt{\frac{2}{\pi}}\,\frac{x}{\sqrt{t}}\,\mathrm{e}^{-\frac{x^2}{2t}} \int_0^1 \Bigl(\int_0^\infty z\mathrm{e}^{-\frac{z^2}{2}} \cosh\bigl(\tfrac{zx}{\sqrt{t}}s\bigr)\mathrm{d}z\Bigr)\mathrm{d}s \end{aligned} \tag{28.43}$$

where we changed variables via $y = z\sqrt{t}$ and then use the symmetry of the integral with respect to s to wrap the result using hyperbolic cosine. For the lower bound in (28.27) we now use that $\cosh(r) \geqslant 1$ while for the upper bound we bound $\cosh(r) \leqslant \mathrm{e}^r$ and then note that

$$\begin{aligned} \mathrm{e}^{-\frac{x^2}{2t}} \int_0^\infty z\mathrm{e}^{-\frac{z^2}{2} + \frac{zx}{\sqrt{t}}}\,\mathrm{d}z &\leqslant \int_0^\infty z\mathrm{e}^{-\frac{(z-x/\sqrt{t})^2}{2}}\mathrm{d}z \\ &\leqslant \int_{-x/\sqrt{t}}^\infty \bigl(u + \tfrac{x}{\sqrt{t}}\bigr)\mathrm{e}^{-\frac{u^2}{2}}\,\mathrm{d}u \leqslant \mathrm{e}^{-\frac{x^2}{2t}} + \sqrt{\frac{\pi}{2}}\,\frac{x}{\sqrt{t}} \end{aligned} \tag{28.44}$$

Bounding the exponential by one, we then get the upper bound in (28.27). The limit statement (28.29) follows from (28.43) and the Dominated Convergence Theorem. □

Further reading: Karatzas-Shreve, Sections 2.6 and 3.5D