23. UNIQUENESS AND LOCALITY

Having furnished criteria for existence of a strong solution, we now address uniqueness. Along with that we also prove locality, which means that the solution depends only on the part of the coefficients that it has"seen" so far.

23.1 Main statement.

An unfortunate feature of Theorem 22.3 is that its assumptions are often too restrictive. Indeed, the setting does not even include the Bessel SDE (which is the only SDE we have seriously considered so far) for the simple fact that the drift term is singular at zero. Another issue is the restriction to square integrable initial conditions which seems to be a technical fact that we should be able to avoid.

Another problem is that the way the solution was constructed does not make it clear whether the part of the solution up to the first exit from a space-time domain depends, as we think it should, only the coefficients $a(t,x)$ and $\sigma(t,x)$ inside this domain. Finally, there is of course the question of uniqueness.

In order to address all of the above issues at once, we put forward:

Definition 23.1 (Strong solution up to a stopping time) *Assume a standard setting for the SDE* (22.7) *with notations as in Definition 22.1. Given a stopping time T for the underlying filtration $\{\mathcal{F}_t\}_{t\geqslant 0}$, we say that $\{X_t\colon t\in[0,T]\}$ is a* strong solution *to* (22.7) up to stopping time T *if*

1. *$\{X_{T\wedge t}\colon t\geqslant 0\}$ is continuous and adapted to $\{\mathcal{F}_t\}_{t\geqslant 0}$,*
2. *for all $t\geqslant 0$,*
$$\int_0^t 1_{\{T>s\}}\bigl|a(s,X_s)\bigr|\mathrm{d}s<\infty\ \wedge\ \int_0^t 1_{\{T>s\}}\bigl|\sigma(s,X_s)\bigr|^2\mathrm{d}s<\infty\quad a.s. \tag{23.1}$$
3. *for all $t\geqslant 0$,*
$$X_{T\wedge t}=X_0+\int_0^t 1_{\{T>s\}}\,a(s,X_s)\mathrm{d}s+\int_0^t 1_{\{T>s\}}\sigma(s,X_s)\cdot\mathrm{d}B_s\quad a.s. \tag{23.2}$$
where X_0 on the right denotes the initial value prescribed in the standard setting.

We can of course take T deterministic, which then defines give the concept of "a strong solution up to a fixed time."

Theorem 23.2 (Uniqueness and locality) *Assume a standard setting with same filtered probability space and Brownian motion, but two pairs of coefficients (a,σ) and $(\tilde a,\tilde\sigma)$ satisfying the Lipschitz-continuity condition* (22.18). *Let $t_0\in(0,\infty]$ and $D\subseteq\mathbb{R}^d$ be non-empty open with*

$$\forall x\in D\,\forall t\in[0,t_0]\cap\mathbb{R}\colon\quad a(t,x)=\tilde a(t,x)\ \wedge\ \sigma(t,x)=\tilde\sigma(t,x) \tag{23.3}$$

If $\{X_t\colon t\in[0,T]\}$ is a strong solution up to (stopping) time T to SDE (22.7) *with coefficients (a,σ) and $\{\widetilde X_t\colon t\in[0,\widetilde T]\}$ is a strong solution up to (stopping) time $\widetilde T$ to SDE* (22.7) *with coefficients $(\tilde a,\tilde\sigma)$ but possibly different initial value, then for*

$$\tau:=t_0\cap T\wedge\widetilde T\wedge\inf\bigl\{t\in[0,T\wedge\widetilde T\colon X_t\notin D\ \vee\ \widetilde X_t\notin D\bigr\} \tag{23.4}$$

we have

$$\forall t \geqslant 0\,\exists C(t) < \infty\colon \quad E\Bigl(\sup_{s\leqslant t\wedge\tau}|X_s-\widetilde{X}_s|^2\Bigr) \leqslant C(t)E\bigl(|X_0-\widetilde{X}_0|^2\bigr) \tag{23.5}$$

In particular,

$$P(X_0=\widetilde{X}_0)=1 \;\Rightarrow\; P\Bigl(\forall t\geqslant 0\colon\; X_{\tau\wedge t}=\widetilde{X}_{\tau\wedge t}\Bigr)=1 \tag{23.6}$$

Proof. The argument is actually very similar to that used in the construction of a strong solution. Indeed, suppose X and $\widetilde{X}$ are two strong solutions and write

$$\tau_r := \tau\wedge\inf\bigl\{t\geqslant 0\colon X_t\notin B(0,r)\;\vee\;\widetilde{X}_t\notin B(0,r)\bigr\} \tag{23.7}$$

where $B(0,r):=\{x\in\mathbb{R}^d\colon |x|<r\}$. Then

$$\forall t\geqslant 0\colon\quad X_{\tau_r\wedge t}-\widetilde{X}_{\tau_r\wedge t}=X_0-\widetilde{X}_0+A_t+M_t \tag{23.8}$$

where, this time,

$$A_t := \int_0^t 1_{\{\tau_r>s\}}\bigl[a(s,X_s)-a(s,\widetilde{X}_s)\bigr]\mathrm{d}s \tag{23.9}$$

and

$$M_t := \int_0^t 1_{\{\tau_r>s\}}\bigl[\sigma(s,X_s)-\sigma(s,\widetilde{X}_s)\bigr]\cdot\mathrm{d}B_s \tag{23.10}$$

which uses that, on $\{\tau_r>s\}$, we have $X_s,\widetilde{X}_s\in D$ and so $\tilde{a}(s,X_s)=a(s,\widetilde{X}_s)$ and $\tilde{\sigma}(s,X_s)=\sigma(s,\widetilde{X}_s)$. Using the inequality $(a+b+c)^2\leqslant 3a^2+3b^2+3c^2$ we get

$$\sup_{s\leqslant t\wedge\tau_r}|X_s-\widetilde{X}_s|^2\leqslant 3|X_0-\widetilde{X}_0|^2+3\sup_{s\leqslant t}A_s^2+3\sup_{s\leqslant t}M_s^2 \tag{23.11}$$

Denoting

$$g_r(t) := E\Bigl(\sup_{s\leqslant t\wedge\tau_r}|X_s-\widetilde{X}_s|^2\Bigr) \tag{23.12}$$

the estimates (22.36–22.37) then show

$$g_r(t)\leqslant 3E\bigl(|X_0-\widetilde{X}_0|^2\bigr)+3K^2(t+4)\int_0^t g_r(s)\mathrm{d}s \tag{23.13}$$

We now invoke:

Lemma 23.3 (Gronwall inquality, simple version) *Let $\gamma\colon[0,\infty)\to[0,\infty)$ be Lebesgue integrable on compact intervals and such that, for some $t_0>0$ and $\alpha,\beta\geqslant 0$,*

$$\forall t\leqslant t_0\colon\quad \gamma(t)\leqslant\alpha+\beta\int_0^t\gamma(s)\,\mathrm{d}s \tag{23.14}$$

Then

$$\forall t\leqslant t_0\colon\quad \gamma(t)\leqslant\alpha\mathrm{e}^{\beta t} \tag{23.15}$$

Proof. Let $\tilde{\alpha} > \alpha$. Then (23.14) and continuity of the integral show $\gamma(t) \leqslant \tilde{\alpha}$ for $t \geqslant 0$ small. The continuity of $t \mapsto \tilde{\alpha}e^{\beta t}$ then gives $t_1 := \sup\{t \in [0,t] \colon \gamma(t) \leqslant \tilde{\alpha}e^{\beta t}\} \in (0,t_0]$. Using this bound in (23.14) shows $\gamma(t) \leqslant \alpha + \tilde{\alpha}[e^{\beta t} - 1]$ for $t \leqslant t_1$ implying, in particular, that $\gamma(t_1) < \tilde{\alpha}e^{\beta t_1}$. If $t_1 < t_0$, then continuity of the integral gives $\gamma(t) < \tilde{\alpha}e^{-\beta t}$ even for t slightly above t_1, contradicting the definition of t_1. So $\gamma(t)e^{-\beta t} \leqslant \tilde{\alpha}$ for all $t \in [0,t_0]$. Taking $\tilde{\alpha} \downarrow \alpha$ we get (23.15). ☐

Using this along with $\tau_r \uparrow \tau$ and the Monotone Convergence Theorem we get

$$\forall t \geqslant 0\colon \quad E\Bigl(\sup_{s \leqslant t \wedge \tau} |X_s - \widetilde{X}_s|^2\Bigr) = \lim_{r\to\infty} g_r(t) \leqslant 3e^{3K^2(t+4)t} E\bigl(|X_0 - \widetilde{X}_0|^2\bigr) \tag{23.16}$$

Setting $C(t) := 3e^{3K^2(t+4)t}$ we get (23.5). For (23.5) we note that $P(X_0 = \widetilde{X}_0) = 1$ implies $g_r(t) = 0$ for all $r < \infty$. Hence $P(\forall t \leqslant \tau_r\colon X_t = \widetilde{X}_t) = 1$. Taking $r \to \infty$ then gives $P(\forall t \leqslant \tau\colon X_t = \widetilde{X}_t) = 1$ as desired. ☐

Theorem 23.2 expresses the intuitive fact that, as long as a solution is generally unique, it will coincide with the solution for another pair of coefficients in the set where the coefficients coincide provided, of course, that solution is started from the same initial value. This is the statement of *locality* for the solutions of SDEs.

We now claim the following extension of Theorems 22.3 and 23.2:

Theorem 23.4 *Assume a standard setting for SDE* (22.7) *with initial value X_0 and let $D \subseteq \mathbb{R}^d$ be non-empty open and such that*

$$P(X_0 \in D) = 1 \tag{23.17}$$

Suppose that there are non-empty bounded open sets $\{D_n\}_{n\geqslant 1}$ with $\forall n \geqslant 1\colon D_n \subseteq D_{n+1}$ and $D = \bigcup_{n\geqslant 1} D_n$ for which the following holds: There is $x_0 \in D_1$ and, for each $n \geqslant 1$, there is $K_n \in (0,\infty)$ such that

$$\forall t \geqslant 0\, \forall x,y \in D_n\colon \quad \bigl|a(t,x) - a(t,y)\bigr| + \bigl|\sigma(t,x) - \sigma(t,y)\bigr| \leqslant K_n|x-y| \tag{23.18}$$

and

$$\forall t \geqslant 0\colon \quad \bigl|a(t,x_0)\bigr| + \bigl|\sigma(t,x_0)\bigr| \leqslant K_n \tag{23.19}$$

Then there exists an a.s. positive stopping time T and a process $\{X_t\colon t \in [0,T)\}$ with initial value X_0 such that, for each $n \geqslant 1$, the process $\{X_{\tau_n \wedge t}\colon t \geqslant 0\}$ is a strong solution to (22.7) *up to stopping time τ_n defined in*

$$\tau_n := \inf\{t \in [0,T)\colon X_t \notin D_n\} \tag{23.20}$$

and

$$T = \lim_{n\to\infty} \tau_n \tag{23.21}$$

Moreover, if $\{\widetilde{X}_t\colon t \in [0,\widetilde{T}]\}$ is another strong solution to (22.7) *up to a stopping time $\widetilde{T}$, then*

$$P(X_0 = \widetilde{X}_0) = 1 \tag{23.22}$$

implies

$$P\Bigl(\forall t \in [0, T \wedge \widetilde{T})\colon X_t = \widetilde{X}_t\Bigr) = 1 \tag{23.23}$$

In short, X is the unique strong solution until the first "exit" from D.

We write the word “exit” in quotes because T may not really be an exit time from D. Indeed, even if T is finite, the solution may simply blow up to infinity at T, or oscillate wildly as t increases to T with no limit value at $t = T$. Notwithstanding, X is continuous on $[0, \tau_n]$ whenever $\tau_n < \infty$. It is these considerations that explain why Definition 23.1 needs to be stated as it is. The condition (23.17) is needed to ensure that $T > 0$ a.s.

That Theorem 23.4 extends Theorems 22.3 and 23.2 is seen by setting $D_n := \mathbb{R}^d$ (and thus also $D = \mathbb{R}^d$) for all $n \geqslant 1$ and noting that then $\tau_n = \infty$ for all $n \geqslant 1$, and thus $T = \infty$, by continuity of the solutions. The solution X will not use the values of the coefficients of the equation for the spatial arguments outside D but $\widetilde{X}$. We bundle the uniqueness clause with the existence of the solution because the uniqueness argument is what actually drives the whole proof.

23.2 Proof of Theorem 23.4.

The proof of Theorem 23.4 is based on the argument that is standard in classical ODE theory: Construct local solutions and then, by showing that they must coincide on their common domain, patch these together to get a maximal solution. For the construction of suitable local solutions — which will be those until the first exit from D_n — we will rely on Theorem 22.3 but for this we need to address the problem that a uniform Lipschitz bound on the coefficients a and σ applies only on D_n. Here we will use:

Lemma 23.5 (Extension of Lipschitz function) *Let $(\mathscr{X}, \rho)$ be a metric space, $A \subseteq \mathscr{X}$ nonempty and $f\colon A \to \mathbb{R}$ a function such that*

$$\forall x, y \in A\colon \quad \bigl|f(x) - f(y)\bigr| \leqslant \rho(x, y) \tag{23.24}$$

Define

$$\forall x \in \mathscr{X}\colon \quad h(x) := \sup_{z \in A}\bigl[f(z) - \rho(x, z)\bigr] \tag{23.25}$$

Then $h(x) \in \mathbb{R}$ for all $x \in \mathscr{X}$ and we have

$$\forall x \in A\colon \quad h(x) = f(x) \tag{23.26}$$

and

$$\forall x, y \in \mathscr{X}\colon \quad \bigl|h(x) - h(y)\bigr| \leqslant \rho(x, y) \tag{23.27}$$

Proof. Since $A \neq \emptyset$, we have $h(x) \in (-\infty, \infty]$. In order to rule out that $h(x)$ is infinite, note that (23.24) along with the triangle inequality imply

$$\forall z, z' \in A\, \forall x \in \mathscr{X}\colon \quad f(z) - \rho(x, z) \leqslant f(z') + \rho(x, z') \tag{23.28}$$

Hence, for each $x \in \mathscr{X}$,

$$h(x) \leqslant \inf_{z \in A}\bigl[f(z) + \rho(x, z)\bigr] \tag{23.29}$$

and so $h(x) < \infty$ as well. (The function defined by the supremum could be another candidate for h.) Since $h(x) < \infty$, for each $\epsilon > 0$ there is $z_\epsilon \in A$ such that $h(x) \leqslant f(z_\epsilon) - \rho(x, z_\epsilon) + \epsilon$. Then for any $y \in \mathscr{X}$, using z_ϵ to get an lower bound on $h(y)$ shows

$$\begin{aligned} h(x) - h(y) &\leqslant f(z_\epsilon) - \rho(x, z_\epsilon) + \epsilon - \bigl[f(z_\epsilon) - \rho(y, z_\epsilon)\bigr] \\ &= \rho(y, z_\epsilon) - \rho(x, z_\epsilon) + \epsilon \leqslant \rho(x, y) + \epsilon \end{aligned} \tag{23.30}$$

where the triangle inequality was used in the last step. Taking $\epsilon \downarrow 0$ and using symmetry between x and y then shows (23.27).

It remains to prove that h is an extension of f. Here the choice $z := x$ for $x \in A$ in (23.25) shows $h(x) \geqslant f(x)$ while taking $z' := x$ in (23.28) and optimizing over $z \in A$ on the left-hand side gives $h(x) \leqslant f(x)$. Hence we get the equality (23.26). $\square$

The key part of the proof of Theorem 23.4 is the content of:

Lemma 23.6 *Assuming the setting of Theorem 23.4, for each $n \geqslant 1$ there exists a process $\{X_t^{(n)} \colon t \geqslant 0\}$ such that (1-3) in Definition 23.1 hold for (X, T) replaced by $(X^{(n)}, T_n)$ where*

$$T_n := n \wedge \inf\{t \geqslant 0 \colon X_t^{(n)} \notin D_n\} \tag{23.31}$$

Moreover, we have

$$\forall n \geqslant 1\colon \quad P\Big(T_n \leqslant T_{n+1} \,\wedge\, \forall t \in [0, T_n]\colon X_t^{(n+1)} = X_t^{(n)}\Big) = 1 \tag{23.32}$$

Proof. For each $t \geqslant 0$ and $x \in \mathbb{R}^d$ define $a^{(n)}(t,x)$ by setting, for each $i = 1, \dots, d$,

$$a_i^{(n)}(t,x) := \sup_{z \in D_n} \big[a_i(t,z) - K_n|x - z|\big] \tag{23.33}$$

Similarly, define $\sigma^{(n)}(t,x)$ by setting, for each $i = 1, \dots, d$ and $j = 1, \dots, m$,

$$\sigma_{ij}^{(n)}(t,x) := \sup_{z \in D_n} \big[\sigma_{ij}(t,z) - K_n|x - z|\big] \tag{23.34}$$

Thanks to Lemma 23.5 and the assumptions (23.18–23.19), $a^{(n)}$ and $\sigma^{(n)}$ now satisfy similar bounds on all of $\mathbb{R}^d$, albeit perhaps with worse constants. Moreover, the continuity of $a(t,\cdot)$ and $\sigma(t,\cdot)$ on D_n permits us to restrict the suprema above to just $z \in D_n \cap \mathbb{Q}^d$ without changing the result. This shows that both $a^{(n)}$ and $\sigma^{(n)}$ are Borel measurable.

Since the boundedness of D ensures that $X_0 1_{\{X_0 \in D\}} \in L^2$, Theorem 22.3 can be applied to construct a strong solution $\{X_t^{(n)} \colon t \geqslant 0\}$ to SDE (22.7) with coefficients $a^{(n)}$ and $\sigma^{(n)}$ and initial value $X_0 1_{\{X_0 \in D\}}$. Explicitly, $X^{(n)}$ is a continuous adapted process such that

$$\forall t \geqslant 0\colon \quad \int_0^t \big|\, a^{(n)}(s, X_s^{(n)})\big| \mathrm{d}s < \infty \,\wedge\, \int_0^t \big|\, \sigma^{(n)}(s, X_s^{(n)})\big|^2 \mathrm{d}s < \infty \quad \text{a.s.} \tag{23.35}$$

and

$$\forall t \geqslant 0\colon \quad X_t^{(n)} = X_0 1_{\{X_0 \in D\}} + \int_0^t a^{(n)}(s, X_s^{(n)})\, \mathrm{d}s + \int_0^t \sigma^{(n)}(s, X_s^{(n)}) \mathrm{d}B_s \quad \text{a.s.} \tag{23.36}$$

Redefine $X^{(n)}$ on $\{X_0 \notin D_n\}$ by setting

$$X_t^{(n)} := X_0 \quad \text{on } \{X_0 \notin D_n\} \tag{23.37}$$

The known properties of the integral then show that $X^{(n)}$ obeys (23.1–23.2).

The processes $X^{(n)}$ and $X^{(n+1)}$ solve the SDE (22.7) with coefficients that coincide in D_n. By Theorem 23.2, these solutions coincide up to the smaller of their first exit times from D_n. This gives (23.32) as desired. $\square$

We are now ready to give:

Proof of Theorem 23.4. Define T_n by (23.31). On the complement of the union of the implicit null sets in (23.32), set

$$\forall t \in [T_{n-1}, T_n)\colon \quad X_t := X_t^{(n)} \tag{23.38}$$

where $T_0 := 0$ for the sake of this definition, and let $X_t := X_0$ otherwise. Thanks to Lemma 23.6, X then obeys (1-3) in Definition 23.1 for each $n \geqslant 1$. In light of the second part of Definition 23.1 and the fact that T_n coincides with τ_n from (23.20), this proves the existence part of the claim.

Concerning uniqueness, let $\{\widetilde{X}_t\colon t \in [0, \widetilde{T}]\}$ be another strong solution up to a stopping time $\widetilde{T}$ with $\widetilde{X}_0 = X_0$ a.s. Since $a^{(n)} = a$ and $\sigma^{(n)} = \sigma$ on D_n, Theorem 23.2 gives

$$P\Big(\forall t \in [0, T_n \wedge \widetilde{T})\colon\ \widetilde{X}_t = X_t^{(n)}\Big) = 1 \tag{23.39}$$

In light of (23.38) and $T_n \uparrow T$, this proves (23.23). □

23.3 Existence of Bessel processes.

As a consequence of Theorem 23.2, we conclude:

Corollary 23.7 *Let $d \in \mathbb{R}$. Then each $X_0 > 0$, the Bessel SDE*

$$\mathrm{d}X_t = \frac{d-1}{2X_t}\mathrm{d}t + \mathrm{d}B_t \tag{23.40}$$

admits a unique strong solution $\{X_t\colon t \in [0, \tau_0)\}$ where

$$\tau_0 := \lim_{\epsilon \downarrow 0} \tau_\epsilon \quad \textit{where} \quad \tau_\epsilon := \inf\{t \geqslant 0\colon |X_t| \leqslant \epsilon\} \tag{23.41}$$

Moreover, if $\tau_0 < \infty$, then this solution extends continuously to $t = \tau_0$ by setting $X_{\tau_0} := 0$.

Proof. The function $x \mapsto \frac{d-1}{2x}$ is uniformly Lipschitz on $D_n := \{x \in \mathbb{R}\colon |x| > 1/n\}$ for each $n \geqslant 1$ and so the solution exists on $D := \bigcup_{n \geqslant 1} D_n = \mathbb{R} \smallsetminus \{0\}$. If $\tau_0 < \infty$ then $X_{\tau_\epsilon} = \epsilon \to 0$ which vanishes in the limit as $\epsilon \downarrow 0$. □

We conclude that a d-dimensional Bessel process exists uniquely for all $d \in \mathbb{R}$ up to the first hitting time of zero beyond which uniqueness can be violated. We will see later that SDE techniques are actually not needed for this conclusion, but this is a different part of the story.

Further reading: Karatzas-Shreve, Section 5.2