21. BESSEL PROCESSES

We proceed to discuss some specific applications of the Itô formula and time change to Brownian motion. The processes we introduce here give us motivation to study stochastic differential equations which we will move to next.

21.1 Definitions.

Recall from (13.58) that if $\mathrm{d}B^{(1)}, \dots, \mathrm{d}B^{(d)}$ are i.i.d. standard Brownian motion, which we interpret as the Cartesian coordinates of a d-dimensional Brownian motion, the radial variable $R_t := [\sum_{i=1}^d (\mathrm{d}B_s^{(i)})^2]^{1/2}$ satisfies the equation

$$\mathrm{d}R_t = \frac{d-1}{2R_t}\mathrm{d}t + \frac{1}{R_t}\sum_{i=1}^d B_t^{(i)}\mathrm{d}B_t^{(i)} \tag{21.1}$$

For the martingale part of this expression we then get:

Lemma 21.1 *Using the notation* (13.56), *the process*

$$\widetilde{B}_t := \sum_{i=1}^d \int_0^t \frac{1}{R_s} B_s^{(i)}\mathrm{d}B_s^{(i)} \tag{21.2}$$

is a standard Brownian motion.

Proof. The quadratic variation process of $\widetilde{B}$ is

$$\langle \widetilde{B} \rangle_t = \sum_{i=1}^d \int_0^t \frac{1}{R_s^2}[B_s^{(i)}]^2\mathrm{d}s = \int_0^t \mathrm{d}s = t \tag{21.3}$$

Theorem 18.1 shows that $\widetilde{B}$ is a standard Brownian motion as claimed. □

In light of this we put forward:

Definition 21.2 *Let* $d \in \mathbb{R}$. *A* d-dimensional Bessel process *is a continuous* $[0,\infty)$-*valued semimartingale* $\{X_t \colon t \geqslant 0\}$ *such that, denoting*

$$\tau_0 := \inf\{t \geqslant 0 \colon X_t = 0\} \tag{21.4}$$

we have $\forall t \geqslant \tau_0 \colon X_t = 0$ *and*

$$\mathrm{d}X_t = \frac{d-1}{2X_t}\mathrm{d}t + \mathrm{d}B_t \quad \text{on } \{\tau_0 > t\} \tag{21.5}$$

where B *is a (one-dimensional) standard Brownian motion.*

The expression (21.5) means that, for each $t \geqslant 0$,

$$X_t = X_0 + \int_0^t 1_{\{\tau_0 > s\}}\frac{d-1}{2X_s}\mathrm{d}s + B_{\tau_0 \wedge t} \tag{21.6}$$

We will write P^x for the law of this process on $C[0,\infty)$ subject to the initial condition

$$P^x(X_0 = x) = 1 \tag{21.7}$$

We note that requirement that X remains constant after hitting 0 for the first time technically makes X a Bessel process with *absorbing boundary condition* at 0. We make this choice as it is convenient in what follows.

Lemma 21.1 combined with the calculation leading to (13.58) show that the radial process of the d-dimensional standard Brownian motion stopped upon first hit of zero is the d-dimensional Bessel process according to Definition 21.2. In particular, a d-dimensional Bessel process exists for all integers $d \geqslant 1$.

A natural question is of course whether a d-dimensional Bessel process exists also for d's that are not positive integers. In addition, we may worry about the process being unique. We will answer both of these questions (affirmatively) using the theory of stochastic differential equations.

21.2 Time a diffusion hits a level set.

Our next question is whether the Bessel process started from $X_0 = x > 0$ hits zero in finite time or not; i.e., whether $\tau_0 < \infty$ or $\tau_0 = \infty$. In order to do this, we will use the following general observation: If X is an $\mathbb{R}$-valued diffusion of the form

$$\mathrm{d}X_t = v(X_t)\mathrm{d}t + \sigma(X_t)\mathrm{d}B_t \tag{21.8}$$

with v and σ continuous such that $\sigma(x) \neq 0$ for all x and $\phi \in C^2(\mathbb{R})$, then the Itô formula ensures that the process $\{\phi(X_t)\colon t \geqslant 0\}$ obeys

$$\mathrm{d}\phi(X_t) = \left[v(X_t)\phi'(X_t) + \frac{1}{2}\phi''(X_t)\sigma(X_t)^2\right]\mathrm{d}t + \phi'(X_t)\sigma(X_t)\mathrm{d}B_t \tag{21.9}$$

Supposing that ϕ satisfies the ODE

$$\frac{1}{2}\phi''(x)\sigma(x)^2 + \phi'(x)v(x) = 0 \tag{21.10}$$

which for given values $\phi(x_0)$ and $\phi'(x_0)$ at some point x_0 is solved explicitly by

$$\phi(x) := \phi(x_0) + \int_{x_0}^{x} \phi'(x_0)\exp\left\{-2\int_{x_0}^{y} \frac{v(z)}{\sigma(z)^2}\mathrm{d}z\right\}\mathrm{d}y \tag{21.11}$$

we get that $\{\phi(X_t)\colon t \geqslant 0\}$ obeys

$$\mathrm{d}\phi(X_t) = \phi'(X_t)\sigma(X_t)\mathrm{d}B_t \tag{21.12}$$

i.e., obeys the necessary condition to be a local martingale. Since the positivity of the exponential also shows that ϕ is strictly monotone (unless it is constant), the question of X hitting a level a is the same as $\phi(X)$ hitting the level $\phi(a)$.

Working with martingales have the advantage that the probability of hitting a given level can be computed explicitly with the help of the Optional Stopping Theorem:

Lemma 21.3 *Let $\hat{a}, \hat{b} \in \mathbb{R} \cup \{\pm\infty\}$ obey $\hat{a} < \hat{b}$ be reals and let X be an $(\hat{a}, \hat{b})$-valued process satisfying* (21.8) *with some $v, \sigma \in C((\hat{a}, \hat{b}))$ such that $\sigma^2 > 0$. Let $\phi \in C^2(\hat{a}, \hat{b})$ be given as in* (21.11) *with $\phi'(x_0) \neq 0$. For each $a \in (\hat{a}, \hat{b})$, set*

$$\tau_a := \inf\{t \geqslant 0\colon X_t = a\} \tag{21.13}$$

Writing P^x for the law of X such that $P^x(X_t = x) = 1$, then for all $a, b \in (\hat{a}, \hat{b})$ with $a < b$,

$$\forall x \in (a,b)\colon \quad P^x(\tau_a \wedge \tau_b < \infty) = 1 \tag{21.14}$$

and

$$\forall x \in (a,b)\colon \quad P^x(\tau_a < \tau_b) = \frac{\phi(b) - \phi(x)}{\phi(b) - \phi(a)} \tag{21.15}$$

(The denominator is non-zero because ϕ is strictly monotone.)

Proof. We start with the proof of (21.14) as it is actually harder and also feeds to the the proof of (21.15). The idea of the proof is to convert the problem that that for standard Brownian motion for which the proof is elementary.

We start with some preliminary considerations. Extending the probability space if necessary, assume that $\widetilde{B}$ is a standard Brownian motion independent of X and B with all three processes adapted to the same filtration $\{\mathcal{F}_t\}_{t \geqslant 0}$. Fix $a, b \in (\hat{a}, \hat{b})$ such that $a < b$ and let $x \in (a, b)$. Assume $\mathcal{F}_0$ contains all P^x-null sets and let Z be a continuous process such that, for each $t \geqslant 0$,

$$Z_t = \phi(X_0) + \int_0^t 1_{\{\tau_a \wedge \tau_b > s\}} \phi'(X_s)\sigma(X_s)\mathrm{d}B_s + 1_{\{\tau_a \wedge \tau_b \leqslant t\}}(\widetilde{B}_t - \widetilde{B}_{\tau_a \wedge \tau_b}) \quad \text{a.s.} \tag{21.16}$$

Since Z is a local martingale with

$$\langle Z \rangle_t = \int_0^t \bigl[1_{\{\tau_a \wedge \tau_b > s\}} \phi'(X_s)^2 \sigma(X_s)^2 + 1_{\{\tau_a \wedge \tau_b \leqslant s\}}\bigr] \mathrm{d}s, \tag{21.17}$$

Letting

$$T(t) := \inf\bigl\{u \geqslant 0 \colon \langle Z \rangle_u \geqslant t\bigr\} \tag{21.18}$$

the fact that $(\phi_d'\sigma)^2$ is bounded and uniformly positive on $[a, b]$ gives

$$\exists c_1, c_2 \in (0, \infty)\, \forall t \geqslant 0\colon \quad c_1 t \leqslant T(t) \leqslant c_2 t \tag{21.19}$$

By Theorem 18.2, $\{Z_{T(t)} \colon t \geqslant 0\}$ is a standard Brownian motion started at $\phi(x)$ under P^x.

Moving to the proof of (21.14), note that the definition of Z gives

$$\forall t \leqslant \tau_a \wedge \tau_b \colon \; Z_t = \phi(X_t) \tag{21.20}$$

The strict monotonicity of ϕ along with (21.19) then imply

$$\begin{aligned} \{\tau_a \wedge \tau_b = \infty\} &= \bigl\{\forall t \geqslant 0 \colon X_t \in (a, b)\bigr\} \\ &= \bigl\{\forall t \geqslant 0 \colon Z_t \in (\phi(a) \wedge \phi(b), \phi(a) \vee \phi(b))\bigr\} \\ &= \bigl\{\forall t \geqslant 0 \colon Z_{T(t)} \in (\phi(a) \wedge \phi(b), \phi(a) \vee \phi(b))\bigr\} \end{aligned} \tag{21.21}$$

The last event involves Brownian motion which, as is readily checked, will not stay confined to bounded interval with positive probability. Hence (21.14) holds.

In order to prove (21.15) we proceed by a familiar argument. Let $M_t := \phi(X_{t \wedge \tau_a \wedge \tau_b})$. Since this process is bounded and obeys

$$\mathrm{d}M_t = 1_{\{\tau_a \wedge \tau_b > t\}} \phi'(X_t)\sigma(X_t)\mathrm{d}B_t \tag{21.22}$$

it is a martingale. Since M is also bounded and, by (21.14), $\tau_a \wedge \tau_b$ is P^x-a.s. finite, the Optional Stopping Theorem (Theorem 18.1(2)) shows

$$\begin{aligned}\phi(x) &= E^x(M_0) = E^x(M_{\tau_a\wedge\tau_b})\\ &= \phi(a)P^x(\tau_a<\tau_b)+\phi(x)P^x(\tau_b<\tau_a)\end{aligned} \tag{21.23}$$

Using that

$$P^x(\tau_b<\tau_a) = 1-P^x(\tau_a<\tau_b) \tag{21.24}$$

by (21.14) again, the formula (21.15) now follows by a simple calculation. □

21.3 Time the Bessel process hits zero.

We now specialize the above general strategy to Bessel processes.

Lemma 21.4 *For $d\in\mathbb{R}$. Then $\phi_d(x)\colon(0,\infty)\to\mathbb{R}$ defined by*

$$\phi_d(x) := \begin{cases} x^{2-d} & \text{if } d\neq 2\\ \log x & \text{if } d=2\end{cases} \tag{21.25}$$

solves the ODE (21.10) *with $v(x):=\frac{d-1}{2x}$ and $\sigma(x):=1$. In particular, if X is a d-dimensional Bessel process and $0<a<b<\infty$, then for all $x\in(a,b)$*

$$\bigl\{\phi_d(X_{\tau_a\wedge\tau_b\wedge t})\colon t\geqslant 0\bigr\} \text{ is a bounded martignale under } P^x \tag{21.26}$$

Proof. That ϕ_d solves the ODE is left to the reader. We will check the martingale property directly as this leads to instructive calculations. Let $\widetilde{X}_t := X_{\tau_a\wedge\tau_b\wedge t}$ and let $\theta\in\mathbb{R}$. The Itô formula then shows

$$\begin{aligned}\mathrm{d}\widetilde{X}_t^\theta &= \theta\widetilde{X}_t^{\theta-1}\mathrm{d}\widetilde{X}_t+\frac12\theta(\theta-1)\widetilde{X}_t^{\theta-2}\mathrm{d}\langle\widetilde{X}\rangle_t\\ &= 1_{\{\tau_a\wedge\tau_b>t\}}\theta X_t^{\theta-1}\mathrm{d}B_t + 1_{\{\tau_a\wedge\tau_b>t\}}\left(\theta\frac{d-1}{2}+\frac12\theta(\theta-1)\right)X_t^{\theta-2}\mathrm{d}t\end{aligned} \tag{21.27}$$

The coefficient of the drift term vanishes when $\theta=2-d$ thus showing that $\{\widetilde{X}_t^{2-d}\colon t\geqslant 0\}$ is a local martingale for all d. This yields a trivial process for $d=2$, so in that case we instead look at

$$\begin{aligned}\mathrm{d}\log\widetilde{X}_t &= \frac{1}{\widetilde{X}_t}\mathrm{d}\widetilde{X}_t-\frac{1}{2\widetilde{X}_t^2}\mathrm{d}\langle\widetilde{X}\rangle_t\\ &= 1_{\{\tau_a\wedge\tau_b>t\}}\frac{1}{X_t}\mathrm{d}B_t+1_{\{\tau_a\wedge\tau_b>t\}}\left(\frac{d-1}{2}+\frac12\right)\frac{1}{X_t^2}\mathrm{d}t\end{aligned} \tag{21.28}$$

and note that here the coefficient of the drift term vanishes exactly when $d=2$. In light of the definition of ϕ_d, this proves the claim. □

Note that the introduction of $\widetilde{X}$ into the calculations in (21.27–21.28) is more or less formal — all it does is to justify the computations via the Itô formula. For this reason, one often invokes the Itô formula directly on the process X with the understanding that the computation is valid only up to the first time the resulting expression develops a singularity. Note that vanishing drift term does not mean that $\phi_d(X)$ is itself a local martingale as that requires finding stopping times $T_n \to \infty$ such that $\phi(X_{T_n \wedge \cdot})$ is a martingale. This must be done by a separate argument.

We are now ready to draw the main conclusions about Bessel processes:

Theorem 21.5 *Let $d \in \mathbb{R}$ and let $\{X_t \colon t \geqslant 0\}$ be a d-dimensional Bessel process started from $x > 0$. Then we have:*

(1) *if $d > 2$, then*

$$\tau_0 = \infty \ \wedge \ \inf_{t \geqslant 0} X_t > 0 \ \wedge \ \limsup_{t \to \infty} X_t = \infty \quad \text{a.s.} \tag{21.29}$$

(2) *if $d < 2$, then*

$$\tau_0 < \infty \ \wedge \ \sup_{t \geqslant 0} X_t < \infty \quad \text{a.s.} \tag{21.30}$$

(3) *if $d = 2$, then*

$$\tau_0 = \infty \ \wedge \ \forall a > 0 \colon \tau_a < \infty \quad \text{a.s.} \tag{21.31}$$

and thus

$$\liminf_{t \to \infty} X_t = 0 \ \wedge \ \limsup_{t \to \infty} X_t = \infty \quad \text{a.s.} \tag{21.32}$$

Proof. We will rely heavily on (21.14–21.15) along with the fact that, regardless of d,

$$\tau_b \underset{b \to \infty}{\longrightarrow} \infty \ \wedge \ \tau_a \underset{a \downarrow 0}{\longrightarrow} \tau_0 \tag{21.33}$$

This follows from the fact that X is continuous which, for the first part, means that the sample paths are locally bounded and, for the second part, that they are uniformly strictly positive on $[0, t]$ for each $t < \tau_0$. We now proceed to deal separately with the cases $d > 2$, $d < 2$ and $d = 2$. We only consider $a, b, x \in (0, \infty)$ such that $0 < a < x < b < \infty$.

Assume first $d > 2$. Then $\phi_d(b) \to 0$ as $b \to \infty$. Using (21.33) and continuity of measure we then get

$$P^x(\tau_a < \infty) = \lim_{b \to \infty} P^x(\tau_a < \tau_b) = \frac{\phi_d(x)}{\phi_d(a)} \tag{21.34}$$

As $\phi_d(a) \to \infty$ as $a \downarrow 0$, this implies

$$\lim_{a \downarrow 0} P^x(\tau_a < \infty) = 0 \tag{21.35}$$

Picking a subsequence $\{a_n\}_{n \geqslant 1}$ with $a_n \downarrow 0$ such that $P^x(\tau_{a_n} < \infty) \leqslant 2^{-n}$, the Borel-Cantelli lemma gives $P(\tau_{a_n} < \infty \text{ i.o}(n)) = 0$ proving $\inf_{t \geqslant 0} X_t > 0$ a.s. In particular, $\tau_0 = \infty$ a.s. and, since $\tau_a = \infty$ for $a < \inf_{t \geqslant 0} X_t$, also that $\tau_b < \infty$ a.s. for all $b > x$, thanks to (21.14). This proves (21.29).

Next let us consider the cases $d < 2$. Then $\phi_d(a) \to 0$ as $a \downarrow 0$ and, by (21.33), we have

$$P^x(\tau_0 < \tau_b) = \lim_{a \downarrow 0} P^x(\tau_a < \tau_b) = 1 - \frac{\phi_d(x)}{\phi_d(b)} \tag{21.36}$$

Taking $b \to \infty$ while noting that, in this case, $\phi_d(b) \to \infty$ as $b \to \infty$ shows

$$P^x(\tau_0 < \infty) = 1 \tag{21.37}$$

But then we must have $\sup_{t \geqslant 0} X_t < \infty$ by continuity of X, proving (21.30).

Turning to the case $d = 2$, here we note that

$$P^x(\tau_a < \tau_b) = \frac{\log b - \log x}{\log b - \log a} \tag{21.38}$$

once $0 < a < x < b < \infty$. The observation (21.33) then gives

$$P^x(\tau_a < \infty) = \lim_{b\to\infty} P^x(\tau_a < \tau_b) = 1 \tag{21.39}$$

and thus also

$$P^x(\tau_0 = \infty) \leqslant \lim_{a\downarrow 0} P^x(\tau_a < \tau_b) = 0 \tag{21.40}$$

It follows that $\tau_c < \infty$ P^x-a.s. for each $c \in (0,\infty)$ regardless of its relation to x. In particular, X is unbounded a.s. which forces $\tau_0 = \infty$ a.s. □

We note that the *limes superior* in (21.29) can be replaced by actual limit. This uses the fact that the Bessel process has the strong Markov property: $\{X_{\tau_b+t} \colon t \geqslant 0\}$ has the law of the Bessel process started at b provided $\tau_b < \infty$ a.s. Taking $b_n := 2^{n^2(d-2)}$ we have

$$P^{b_n}(\tau_{b_{n+1}} < \tau_{b_{n-1}}) = \frac{b_n^{2-d} - b_{n-1}^{2-d}}{b_{n+1}^{2-d} - b_{n-1}^{2-d}} \leqslant \frac{1}{4^n - 1} \tag{21.41}$$

which is summable on $n \geqslant 1$. A Borel-Cantelli argument then shows that X will not hit b_n and then return to b_{n-1} more than for finitely many $n \geqslant 1$. Hence, X is eventually larger than b_{n-1} for all n large and since $b_n \to \infty$, also $X_t \to \infty$ a.s. Note that this conclusion fails for $d = 2$.

An obvious technical advantage of the cases with $d \geqslant 2$ is that the stochastic differential equation (21.5) applies to all times. The cases $d < 2$ will serve as an example of a solution to a stochastic differential equation that hits a singularity in finite time a.s.

21.4 Recurrence and transience for Brownian motion.

Thanks to fact that the radial process of a d-dimensional standard Brownian motion is a d-dimensional Bessel process, Theorem 21.5 now gives:

Corollary 21.6 (Recurrence/transience of Brownian motion) *For d-dimensional standard Brownian motion B, the following holds:*

(1) *in $d = 1$, B is recurrent to points, meaning that*

$$\{B_t \colon t \geqslant 0\} = \mathbb{R} \quad \text{a.s.} \tag{21.42}$$

(2) *in $d = 2$, B is not recurrent to points but is recurrent to open balls, meaning that*

$$\forall x \in \mathbb{R}^2 \smallsetminus \{0\}\colon \quad P(\exists t \geqslant 0\colon B_t = x) = 0 \tag{21.43}$$

yet

$$\{B_t \colon t \geqslant 0\} \text{ is dense in } \mathbb{R}^2 \quad \text{a.s.} \tag{21.44}$$

(3) *in $d \geqslant 3$, B is not even recurrent to open balls, meaning that*

$$\forall x \in \mathbb{R}^d \smallsetminus \{0\}\colon \quad \inf_{t\geqslant 0} |B_t - x| > 0 \quad \text{a.s.} \tag{21.45}$$

Proof. In $d = 1$, Theorem 21.5 along with the symmetries of the Brownian motion give $P^x(\tau_a < \infty) = 1$ for all $a \in \mathbb{R}$. Hence $P^x(\forall n \in \mathbb{Z}\colon \tau_n < \infty) = 1$. The latter event implies that in (21.42) by continuity of B.

In $d = 2$, Theorem 21.5 shows $\{R_t\colon t \geqslant 0\} = (0, \infty)$ P^x-a.s. for each $x \neq 0$. This means that, for Brownian motion started at x, a.e. path will not visit any fixed $x \neq 0$ a.s. yet comes arbitrarily close to every $x \in \mathbb{Q}^2$. In particular, a.e. path is dense in $\mathbb{R}^2$.

Finally, in $d \geqslant 3$, Theorem 21.5 shows that $\inf_{t\geqslant 0} R_t > 0$ P^x-a.s. for each $x \neq 0$. This directly translates to (21.45). □

Notice that the case $d = 2$ is formally distinct from that of two-dimensional simple random walk, which is recurrent to points in $d = 2$. But the recurrence is just marginal; indeed, the number of returns in time n is order $\log n$. (For one dimensional walk it is order $\sqrt{n}$.) We also get another interesting fact about two-dimensional Brownian paths:

Corollary 21.7 *Let $d \geqslant 2$. Then almost every path of the d-dimensional Brownian motion is a $\mathcal{F}_\sigma$-set of vanishing Lebesgue measure (in spite of being dense in $\mathbb{R}^2$ when $d = 2$).*

We leave the proof (which is mainly an exercise in measurability) to a homework problem for the reader.

Further reading: Karatzas-Shreve, Section 3.3C