20. ADDITIVE CHAOS THEORY

Here we introduce iterated Itô integrals and, for a specific kind of non-random integrands, prove that every square integrable random variable depending only on a portion of the path of the Brownian motion can be expanded into an infinite series of these integrals. This representation is usually called the chaos expansion.

20.1 Iterated Itô integrals.

An iterated Itô integral is generally an expression of the form

$$\int_0^t \Bigl(\int_0^s Z_u \mathrm{d}M_u\Bigr) Y_s \mathrm{d}N_s \tag{20.1}$$

where $M, N \in \mathscr{M}_{\text{loc}}^{\text{cont}}$, $Z \in \mathcal{V}_M^{\text{loc}}$ and $Y \in \mathcal{V}_N^{\text{loc}}$ and where we assume that all integrals represent their continuous versions. The local boundedness of continuous functions ensures that the outer integral exists just under the assumption $Y \in \mathcal{V}_N^{\text{loc}}$.

Higher order iterated integrals can of course be formed by taking Z that is itself an Itô integral. However, unlike ordinary integrals, iterated Itô integrals are unwieldy due to the fact that the order of integration cannot generally be interchanged due to adaptedness requirements. For the same reason we cannot generally consider integrands that depend on both integrated variables. However, these issues go away when the integrand is non-random which is what we will focus from now on. We will also work primarily with integrals with respect to Brownian motion.

Throughout this section, an iterated integral is thus an expression of the form

$$\int_0^t \Bigl(\int_0^{t_n} \Bigl(\dots \Bigl(\int_0^{t_2} f(t_1,\dots,t_n)\mathrm{d}B_{t_1}\Bigr) \dots \Bigr)\mathrm{d}B_{t_{n-1}}\Bigr)\mathrm{d}B_{t_n} \tag{20.2}$$

where $f\colon [0,\infty)^n \to \mathbb{R}$ is a function with suitable measurability and integrability properties. To define this precisely, we start with some notations.

Noting that the integral (20.2) "sees" only the arguments of f where $t_1 < t_2 \cdots < t_n$, the function really only needs to be defined on the set

$$D_n := \bigl\{(t_1,\dots,t_n) \in \mathbb{R}^n \colon 0 \leqslant t_1 < t_2 < \dots < t_n\bigr\} \tag{20.3}$$

We endow D_n with the n-dimensional Lebesgue measure and write $L^{2,\text{loc}}(D_n)$ for the space of locally square-integrable functions $f\colon D_n \to \mathbb{R}$. For $f\colon D_n \to \mathbb{R}$ and $t \geqslant 0$, let $f_t\colon D_{n-1} \to \mathbb{R}$ denote the function

$$f_t(t_1,\dots,t_{n-1}) := 1_{\{t_{n-1}<t\}} f(t_1,\dots,t_{n-1},t) \tag{20.4}$$

We will henceforth assume existence of a probability space supporting a Brownian motion B adapted to the filtration $\{\mathcal{F}_t\}_{t\geqslant 0}$ with $\mathcal{F}_0$ containing all P-null sets. Recall also that $\mathcal{V}_B$ is the space of adapted, jointly-measurable processes $\{Y_s \colon s \geqslant 0\}$ such that $s,\omega \mapsto Y_s(\omega)$ is in $L^2([0,t]\times\Omega)$ for all $t \geqslant 0$. We then have:

Proposition 20.1 (Iterated Itô integrals) *For all $n \geqslant 1$ and all $f \in L^{2,\text{loc}}(D_n)$ there exists a continuous L^2-martingale $\{I_t^{(n)}(f)\colon t \geqslant 0\}$ with $I_0^{(n)}(f) = 0$ such that the following holds:*

(1) *If* $n = 1$, *then*

$$\forall f \in L^{2,\mathrm{loc}}(D_1)\,\forall t \geqslant 0\colon\;\; I_t^{(1)}(f) = \int_0^t f(s)\mathrm{d}B_s \quad \text{a.s.} \tag{20.5}$$

(2) *If* $n \geqslant 2$ *and* $f \in L^2(D_n)$, *then there exists* $Y \in \mathcal{V}_B$ *with the property that*

$$\forall t \geqslant 0\colon \quad Y_t = I_t^{(n-1)}(f_t) \quad \text{a.s.} \tag{20.6}$$

and

$$\forall t \geqslant 0\colon \quad I_t^{(n)}(f) = \int_0^t Y_s\,\mathrm{d}B_s \quad \text{a.s.} \tag{20.7}$$

Moreover, for each $n \geqslant 1$ *and* $t \geqslant 0$, *the map* $f \mapsto I_t^{(n)}(f)$ *obeys*

$$E\big(I_t^{(n)}(f)^2\big) = \int_{D_n \cap [0,t]^n} f(t_1,\dots,t_n)^2\,\mathrm{d}t_1\dots\mathrm{d}t_n \tag{20.8}$$

and so defines a linear isometry $L^2(D_n \cap [0,t]^n) \to L^2(\Omega, \mathcal{F}^B, P)$.

Before we delve into the proof, note that the above statements allow us to define the object in (20.2) as

$$\int_0^t\Big(\int_0^{t_n}\Big(\dots\Big(\int_0^{t_2} f(t_1,\dots,t_n)\mathrm{d}B_{t_1}\Big)\dots\Big)\mathrm{d}B_{t_{n-1}}\Big)\mathrm{d}B_{t_n} := I_t^{(n)}(f). \tag{20.9}$$

Indeed, (20.6–20.7) express the nesting property of these integrals which is intuitive but, since these are not ordinary integrals, has to be handled with care. We state the nesting this way because we do not want to deal with the regularity of $t \mapsto I_t^{(n-1)}(f_t)$. This explanation should be enough for us to get into:

Proof of Proposition 20.1. We proceed as in the construction of the Itô integral. First, let us call $f\colon D_n \to \mathbb{R}$ simple if for some $m \geqslant n$ and $0 \leqslant s_1 < \dots < s_m$ and some collection of numbers $\{a_{j_1,\dots,j_n}\colon 1 \leqslant j_1 < j_2 < \dots < j_n \leqslant m\} \subseteq \mathbb{R}$,

$$f(t_1,\dots,t_n) = \sum_{1\leqslant j_1<j_2<\dots<j_n\leqslant m} a_{j_1,\dots,j_n}\prod_{k=1}^n 1_{(s_{j_k-1},s_{j_k}]}(t_k) \tag{20.10}$$

holds for all $(t_1,\dots,t_n) \in D_n$. We then define $t \mapsto I_t^{(n)}(f)$ as

$$I_t^{(n)}(f) := \sum_{1\leqslant j_1<j_2<\dots<j_n\leqslant m} a_{j_1,\dots,j_n}\prod_{k=1}^n \big(B_{s_{j_k}\wedge t} - B_{s_{j_k-1}\wedge t}\big) \tag{20.11}$$

which requires checking (left to the reader) that the right-hand side does not depend on the representation of f as above.

Note that for $n = 1$ this is exactly the definition of the Itô integral of a simple function. Observe also that, for $n \geqslant 2$, we have

$$I_t^{(n-1)}(f_t) = \sum_{1\leqslant j_1<j_2<\dots<j_n\leqslant m} a_{j_1,\dots,j_n} 1_{(s_{j_n-1},s_{j_n}]}(t)\prod_{k=1}^{n-1}\big(B_{s_{j_k}\wedge t} - B_{s_{j_k-1}\wedge t}\big) \tag{20.12}$$

which is checked to be adapted and piecewise constant (as a process indexed by t). Setting $Y_t := I_t^{(n-1)}(f_t)$ we observe that Y has the form of a simple process, except for the boundedness requirement. A direct integration gives (20.7). The isometry and the continuous-martingale property are then checked readily as well.

The claim thus holds for f simple so the main piece of work is to extend it to all $f \in L^{2,\mathrm{loc}}(D_n)$. For this we observe:

Lemma 20.2 *Simple functions of the form* (20.10) *are dense in* $L^{2,\mathrm{loc}}(D_n)$.

Proof. It suffices to show that if $h \in L^2(D_n)$ with compact support is orthogonal to all simple functions, then $h = 0$. Given $0 \leqslant a_1 < b_1 \leqslant a_2 < b_2 \leqslant \dots \leqslant a_n < b_n$, the function

$$f(t_1, \dots, t_n) := \prod_{i=1}^n 1_{(a_i,b_i]}(t_i) \tag{20.13}$$

is simple. The orthogonality of h to f then shows that the integral of h vanishes on all sets of the form $\times_{i=1}^n (a_i, b_i]$ for $\{a_i, b_i\}_{i=1}^n$ as above. But these sets form a semialgebra (and thus a π-system) that generates all Borel subsets of D_n. Fix any $t \geqslant 0$. Since the class of Borel subsets of $D_n \cap [0,t]^n$ on which the integral of h vanishes forms a λ-system, Dynkin's π/λ-Theorem shows that the integral of h vanishes on all Borel subsets of $D_n \cap [0,t]^n$. This implies $h = 0$ Lebesgue a.e. as desired. □

Continuing the proof of Proposition 20.1, Lemma 20.2 allows us to approximate any $f \in L^{2,\mathrm{loc}}(D_n)$ by a sequence of simple functions $\{f^{(k)}\}_{k\geqslant 1}$ so that

$$\|f - f^{(k)}\|_{L^{2,\mathrm{loc}}(D_n)} \leqslant 4^{-k} \tag{20.14}$$

for each $k \geqslant 1$. Noting that (20.8) for $n-1$ instead of n gives

$$\int_0^t E\Bigl(\bigl[I_s^{(n-1)}(f_s^{(k+1)}) - I_s^{(n-1)}(f_s^{(k)})\bigr]^2\Bigr)\mathrm{d}s \leqslant \|f^{(k+1)} - f^{(k)}\|^2_{L^{2,\mathrm{loc}}(D_n)} \leqslant 16^{1-k} \tag{20.15}$$

the Markov inequality shows

$$P\biggl(\lambda\Bigl(\bigl\{s \in [0,t] \colon |I_s^{(n-1)}(f_s^{(k+1)}) - I_s^{(n-1)}(f_s^{(k)})| > 2^{-k}\bigr\}\Bigr) > 2^{-k}\biggr) \leqslant 8^k 16^{1-k} \tag{20.16}$$

Write $\Omega^\star$ for the event that the event in the probability above occurs only for finitely-many k. Setting

$$Y_t := \limsup_{k\to\infty} I_t^{(n-1)}(f_t^{(k)}) 1_{\{\limsup_{k\to\infty} I_t^{(n-1)}(f_t^{(k)}) \in \mathbb{R}\}} \tag{20.17}$$

on $\Omega^\star$ and $Y_t := 0$ on $\Omega \smallsetminus \Omega^\star$, then the limit in

$$Y_t := \lim_{k\to\infty} I_t^{(n-1)}(f_t^{(k)}) \tag{20.18}$$

exists and equality holds for Lebesgue a.e. $t \in [0, \infty)$ on $\Omega^\star$.

The Borel-Cantelli lemma implies that $P(\Omega^\star) = 1$ and the fact $\mathcal{F}_0$ contains all P-null sets gives $\Omega^\star \in \mathcal{F}_0$. The process Y is then jointly measurable and adapted. The inequality

(20.15) with the help of Fatou's lemma also shows that

$$\int_0^t E\Big(\big[I_s^{(n-1)}(f_s^{(k)}) - Y_s\big]^2\Big) \underset{k\to\infty}{\longrightarrow} 0 \tag{20.19}$$

giving us (20.6). The convergence also implies $Y \in \mathcal{V}_B$ and the Itô integral in (20.7) is well defined and equal to the L^2-limit of the integrals in

$$I_t^{(n)}(f^{(k)}) = \int_0^t I_s^{(n-1)}(f_s^{(k)})\mathrm{d}B_s. \tag{20.20}$$

Using again that $\mathcal{F}_0$ contains all P-null sets, we now define $\{I_t^{(n)}(f)\colon t \geqslant 0\}$ to be a continuous version of $\{\int_0^t Y_s \mathrm{d}B_s\colon t \geqslant 0\}$. Then (20.7) holds and the process is a continuous L^2-martingale as claimed. Validating also the isometry (20.8) by extension from simple functions, the proof is finished. □

20.2 Chaos expansion.

We will henceforth write $I_t^{(n-1)}(f_t)$ for the process $Y \in \mathcal{V}_B$ such that (20.6–20.7) holds. Whenever these are invoked, the reader should remember that these are a.s./L^2-limits of actual integrals of f_t for f simple. Here is a key property of iterated Itô integrals:

Lemma 20.3 *For all $m, n \geqslant 1$, $f \in L^2(D_n \cap [0,t]^n)$, $g \in L^2(D_m \cap [0,t]^m)$ and $t, u \geqslant 0$,*

$$m \neq n \quad \Rightarrow \quad E\Big(I_t^{(n)}(f) I_u^{(m)}(g)\Big) = 0 \tag{20.21}$$

In particular, iterated Itô integrals with distinct number of variables are orthogonal in L^2.

Proof. Using the facts about single-variate Itô integrals we have

$$E\Big(I_t^{(n)}(f) I_u^{(m)}(g)\Big) = \int_0^{t\wedge u} E\Big(I_s^{(n-1)}(f_s) I_s^{(m-1)}(g_s)\Big)\mathrm{d}s \tag{20.22}$$

The right-hand side vanishes for any $n > 1$ and $m = 1$ by the fact that the Itô integral is centered. Proceeding inductively, the claim follows for any $n > m$. □

Using this observation, we now get to main result of this section:

Theorem 20.4 *Fix $t > 0$ and let $B = \{B_s\colon s \in [0,t]\}$ be a standard Brownian motion defined on some probability space $(\Omega, \mathcal{F}, P)$. Let $\widetilde{\mathcal{F}}_t^B := \sigma(\mathcal{N} \cup \sigma(B_s\colon s \leqslant t))$ for $\mathcal{N} := \{P\text{-null sets}\}$. Let $\mathcal{H}_0$ be the set of constant random variables on $(\Omega, \widetilde{\mathcal{F}}_t^B, P)$ and, for each $n \geqslant 1$, set*

$$\mathcal{H}_n := \Big\{I_t^{(n)}(f)\colon f \in L^2(D_n \cap [0,t]^n)\Big\} \tag{20.23}$$

Then $\{\mathcal{H}_n\}_{n\geqslant 0}$ are (if regarded as a collection of equivalence classes of random variables) orthogonal closed linear subspaces of $L^2(\Omega, \widetilde{\mathcal{F}}_t^B, P)$ and we have

$$L^2(\Omega, \widetilde{\mathcal{F}}_t^B, P) = \bigoplus_{n\geqslant 0} \mathcal{H}_n \tag{20.24}$$

Proof. Thanks to the isometry (20.8) (and the construction of iterated Itô integrals by L^2-limits of those of simple processes), $\mathcal{H}_n$ is a closed linear subspace of $L^2(\Omega, \widetilde{\mathcal{F}}_t^B, P)$. Lemma 20.3 implies $\mathcal{H}_n \perp \mathcal{H}_m$ whenever $m \neq n$. All we have to do is that linear combinations of iterated Itô integrals up to time t (and constants) are dense in $L^2(\Omega, \widetilde{\mathcal{F}}_t^B, P)$.

For $m \geqslant n$ let $0 = s_0 < s_1 < \cdots < s_m = t$ and for $1 \leqslant j_1 < \cdots < j_n \leqslant m$ define

$$f(t_1, \dots, t_n) := \prod_{k=1}^n 1_{(s_{j_k-1}, s_{j_k}]}(t_k) \tag{20.25}$$

Then

$$I_t^{(n)}(f) = \prod_{k=1}^n (B_{s_{j_k}} - B_{s_{j_k-1}}) \tag{20.26}$$

For any $\ell = 1, \dots, m$ the usual telescopic trick gives

$$B_{s_\ell} I_t^{(n)}(f) = \sum_{i=1}^{\ell} (B_{s_i} - B_{s_{i-1}}) \prod_{k=1}^n (B_{s_{j_k}} - B_{s_{j_k-1}}) \tag{20.27}$$

If i is not part of the sequence $j_1, \dots, j_n$, then the left hand side is the iterated integral of another function of the kind (20.25), albeit now with $n+1$ variables. If i is part of the sequence $j_1, \dots, j_n$, then one of the terms gets squared. Here we use that, by the Itô formula, for any $s \leqslant t$,

$$\begin{aligned}(B_t - B_s)^2 &= (t-s) + 2\int_s^t B_u \mathrm{d}B_u \\ &= (t-s) + 2\int_0^t \Big(\int_0^{t_2} 1_{(s,t_2)}(t_1) 1_{(s,t]}(t_2) \mathrm{d}B_{t_1} \Big) \mathrm{d}B_{t_2}\end{aligned} \tag{20.28}$$

This shows that

$$\Bigg(\prod_{\substack{k=1,\dots,n \\ k \neq \ell}} (B_{s_{j_k}} - B_{s_{j_k-1}}) \Bigg) (B_{s_{j_\ell}} - B_{s_{j_\ell-1}})^2 = I_t^{(n+1)}(g) + I_t^{(n-1)}(h) \tag{20.29}$$

for

$$g(t_1, \dots, t_{n-1}) := (s_{j_\ell} - s_{j_\ell-1}) \Bigg(\prod_{k=1}^{\ell-1} 1_{(s_{j_k-1}, s_{j_k}]}(t_k) \Bigg) \Bigg(\prod_{k=\ell+1}^{n} 1_{(s_{j_k-1}, s_{j_k}]}(t_{k-1}) \Bigg) \tag{20.30}$$

and

$$\begin{aligned} &h(t_1, , \dots, t_{n+1}) \\ &= 2\Bigg(\prod_{k=1}^{\ell-1} 1_{(s_{j_k-1}, s_{j_k}]}(t_k) \Bigg) 1_{(s_{j_\ell-1}, t_{\ell+1})}(t_\ell) 1_{(s_{j_\ell-1}, s_{j_\ell}]}(t_{\ell+1}) \Bigg(\prod_{k=\ell+1}^{n} 1_{(s_{j_k-1}, s_{j_k}]}(t_{k+1}) \Bigg) \end{aligned} \tag{20.31}$$

We thus get that, for all $n \geqslant 1$ and all $f \in L^2(D_n \cap [0,t]^n$,

$$f \text{ simple} \quad \Rightarrow \quad \forall s \in [0,t] \colon \; B_s I_t^{(n)}(f) \in \bigoplus_{k=0}^{n+1} \mathcal{H}_k \tag{20.32}$$

If $f \in L^2(D_n \cap [0,t]^n)$ is general, then we can find simple $f^{(k)}$ such that $f^{(k)} \to f$ in $L^2(D_n \cap [0,t]^n)$. Then $I_t^{(n)}(f^{(k)}) \to I_t^{(n)}(f)$ in L^2. We would like to claim that also $B_s I_t^{(n)}(f^{(k)}) \to B_s I_t^{(n)}(f)$ in L^2 but, since B_s is not bounded, this requires some higher moment information. This is supplied in:

Lemma 20.5 *For all $n \geqslant 1$, all $t \geqslant 0$ and all $f \in L^{2,\mathrm{loc}}(D_n)$,*

$$E\big(I_t^{(n)}(f)^4\big) \leqslant 6^{2n}\left(\int_{D_n \cap [0,t]^n} f(t_1,\dots,t_n)^2\right)^2 \tag{20.33}$$

In particular, the map $f \mapsto I_t^{(n)}(f)$ is continuous as a map $L^{2,\mathrm{loc}}(D_n) \to L^4$.

Postponing the proof until the present proof is finished, hereby we get that $f^{(k)} \to f$ in $L^2(D_n \cap [0,t]^n)$ actually implies $I_t^{(n)}(f^{(k)}) \to I_t^{(n)}(f)$ in L^4. Using the Cauchy-Schwarz inequality, this now readily gives $B_s I_t^{(n)}(f^{(k)}) \to B_s I_t^{(n)}(f)$ in L^2, proving that

$$\forall s \in [0,t]\colon\ B_s I_t^{(n)}(f) \in \bigoplus_{k=0}^{n+1} \mathcal{H}_k \tag{20.34}$$

for all $f \in L^{2,\mathrm{loc}}(D_n)$ relying also on the fact that $\bigoplus_{k=0}^{n+1} \mathcal{H}_k$ is closed.

Since $B_s \in \mathcal{H}_1$ by the fact that $B_s = I_t^{(1)}(1_{[0,s]})$, from (20.34) we inductively obtain that, for all $k \geqslant 1$, all $0 \leqslant t_1 < \dots < t_k \leqslant t$ and all $n_1,\dots,n_k \geqslant 1$,

$$B_{t_1}^{n_1}\dots B_{t_k}^{n_k} \in \bigoplus_{k=0}^{n} \mathcal{H}_k \quad \text{where} \quad n := \sum_{i=1}^{k} n_i \tag{20.35}$$

By linearity, all polynomials in individual values of B lie in $\bigoplus_{n\geqslant 0} \mathcal{H}_n$. But polynomials in a finite number of random variables are dense in L^2-space of Borel functions of these variables. Specializing to indicators, for each $X \in L^2(\Omega, \widetilde{\mathcal{F}}_t^B, P)$ we conclude

$$E\big(X \,\big|\, \sigma(B_{t_1},\dots,B_{t_n})\big) \in \bigoplus_{n\geqslant 0} \mathcal{H}_n \tag{20.36}$$

Taking a sequence $\{t_i\}_{i\geqslant 1}$ exhausting $\mathbb{Q} \cap [0,t]$ and invoking the Lévy Forward Theorem along with the fact that $\sigma(B_s\colon \mathbb{Q} \cap [0,t]) = \mathcal{F}_t^B$ we get $X \in \bigoplus_{n\geqslant 0} \mathcal{H}_n$ as desired. □

It remains to give:

Proof of Lemma 20.5. We begin by showing that, for any $M \in \mathscr{M}_{\mathrm{loc}}^{\mathrm{cont}}$ and all $t \geqslant 0$,

$$E(M_t^4) \leqslant 36 E\big(\langle M\rangle_t^2\big) \tag{20.37}$$

(This is a simple version of powerful martingale moment inequalities whose ultimate form was isolated by Burkholder, Davis and Gundy; see Theorem 3.28 in Karatzas-Shreve.) Using a stopping time argument, it suffices to prove the inequality under the assumption that M and $\langle M\rangle$ are bounded. Since the Itô formula gives

$$M_t^4 = \int_0^t 4M_s^3 \mathrm{d}M_s + \int_0^t 6M_s^2 \mathrm{d}\langle M\rangle_s \tag{20.38}$$

we have

$$E(M_t^4) = 6E\biggl(\int_0^t M_s^2 \mathrm{d}\langle M\rangle_s\biggr) \tag{20.39}$$

The integral on the right is in Stieltjes sense, so we can manipulate it as follows: rewrite the integrand as $M_s^2 = \langle M\rangle_s + M_s^2 - \langle M\rangle_s$ and then observe that, by the fact that $M^2 - \langle M\rangle$ is a martingale,

$$E\biggl(\int_0^t (M_s^2 - \langle M\rangle_s)\mathrm{d}\langle M\rangle_s\biggr) = E\biggl(\int_0^t (M_t^2 - \langle M\rangle_t)\mathrm{d}\langle M\rangle_s\biggr) = E\bigl(M_t^2\langle M\rangle_t\bigr) - E\bigl(\langle M\rangle_t^2\bigr) \tag{20.40}$$

(Indeed, approximate the integral by a right-endpoint Riemann sum, for which this is checked directly.) Applying $\langle M\rangle_s \leqslant \langle M\rangle_t$ in the integral $\int_0^t \langle M\rangle_s \mathrm{d}\langle M\rangle_s$ then yields

$$E(M_t^4) \leqslant 6E\bigl(M_t^2\langle M\rangle_t\bigr) \tag{20.41}$$

Invoking the Cauchy-Schwarz inequality then gives (20.37).

With the help of (20.37), the nesting property of the iterated Itô integrals and Tonelli's theorem we now get

$$E\bigl(I_t^{(n)}(f)^4\bigr) \leqslant 36 \int_{[0,t]^2} E\bigl(I_s^{(n-1)}(f_s)^2 I_u^{(n-1)}(f_u)^2\bigr)\mathrm{d}s\mathrm{d}u \tag{20.42}$$

which after invoking the Cauchy-Schwarz inequality under the expectation shows

$$E\bigl(I_t^{(n)}(f)^4\bigr) \leqslant 36 \biggl[\int_{[0,t]} E\bigl(I_s^{(n-1)}(f_s)^4\bigr)^{1/2}\mathrm{d}s\biggr]^2 \tag{20.43}$$

Since $I_s^{(0)}(f_s) = f(s)$, this proves the claim for $n = 1$. Assuming that the claim holds for $n-1$, we have

$$E\bigl(I_s^{(n-1)}(f_s)^4\bigr)^{1/2} \leqslant 6^{n-1} \int_{D_{n-1}\cap[0,t]^{n-1}} f(t_1,\dots,t_{n-1},s)1_{\{t_{n-1}<s\}}\mathrm{d}t_1\dots\mathrm{d}t_{n-1} \tag{20.44}$$

Integrating the right-hand side with respect to s then yields the claim for n. □

We remark that the continuity of $f \mapsto I_t^{(n)}(f)$ as a map $L^{2,\mathrm{loc}}(D_n) \to L^p$ can be shown for all $p \in [1,\infty)$ albeit with a different constant and a more complicated proof.

20.3 Some consequences.

An element of $\mathcal{H}_n$ is sometimes called a *chaos of order n*. The statement of Theorem 20.4 can be paraphrased as an expansion to chaos of all orders:

Corollary 20.6 (Chaos expansion) *For each $t \geqslant 0$ and each $X \in L^2(\Omega, \widetilde{\mathcal{F}}_t^B, P)$ there exist $\{h^{(k)}\}_{k\geqslant 1}$ with $h^{(k)} \in L^2(D_k \cap [0,t]^k)$ for each $k \geqslant 1$ such that*

$$X = EX + \sum_{k\geqslant 1} I_t^{(k)}(h^{(k)}) \tag{20.45}$$

with the sum convergent in L^2.

Proof. The projection of X onto $\mathcal{H}_0$ equals EX. The claim is then just a restatement of (20.24) using explicit functions. □

The chaos expansion offers a different proof of some of the earlier representation results that were obtained using the "single-variable" stochastic calculus. The following already appeared as Theorem 19.1:

Corollary 20.7 *Let $B = \{B_s : s \leqslant t\}$ be a standard Brownian motion on $(\Omega, \mathcal{F}, P)$. Then for any $t \geqslant 0$ and any $X \in L^2(\Omega, \widetilde{\mathcal{F}}_t^B, P)$, where $\widetilde{\mathcal{F}}_t^B$ is as above, there is $Y \in \mathcal{V}_B$ such that*

$$X = EX + \int_0^t Y_s \,\mathrm{d}B_s \tag{20.46}$$

Proof. Using the same notation as the statement of Corollary 20.6, we have

$$X = EX + \sum_{k\geqslant 1} \int_0^t I_s^{(k-1)}(h_s^{(k)})\mathrm{d}B_s \tag{20.47}$$

with the elements orthogonal and the sum convergent in L^2. The Itô isometry then ensures that the sum in

$$Y_s := 1_{[0,t]}(s) \sum_{k\geqslant 1} I_s^{(k-1)}(h_s^{(k)}) \tag{20.48}$$

converges in $L^2(\Omega \times [0,t])$ and so the sum in (20.47) can be exchanged with the Itô integral. The claim follows from the nesting property (20.6–20.7). □

Just as in the previous section, Corollary 20.7 now yields also Theorem 19.2 representing Brownian martingales as stochastic integrals with respect to Brownian motion. An interesting question is what this representation looks like for some simple martingales; e.g., B_t, $B_t^2 - t$, etc. These martingales are produced by expanding the exponential martingale $\mathrm{e}^{\lambda B_t - \frac{1}{2}\lambda^2 t}$ into a power series in λ. So, generalizing slightly, we first give:

Lemma 20.8 *Let $t \geqslant 0$ and let $f \in L^2([0,t])$. Then*

$$\exp\left\{\int_0^t f(s)\mathrm{d}B_s - \frac{1}{2}\int_0^t f(s)^2\mathrm{d}s\right\} = 1 + \sum_{n\geqslant 1} I_t^{(n)}\big(1_{D_n} f^{\otimes n}\big) \quad \text{a.s.} \tag{20.49}$$

where $f^{\otimes n} : \mathbb{R}_+^n \to \mathbb{R}$ is defined by

$$f^{\otimes n}(t_1, \dots, t_n) := \prod_{i=1}^n f(t_i) \tag{20.50}$$

and D_n is as in (20.3). *The sum in* (20.49) *converges in L^2.*

We leave the proof of this lemma to homework. Hence we get:

Corollary 20.9 *For all $t \geqslant 0$ and $\lambda \in \mathbb{R}$,*

$$\mathrm{e}^{\lambda B_t - \frac{1}{2}\lambda^2 t} = 1 + \sum_{n\geqslant 1} \lambda^n \int_0^t \bigg(\int_0^{t_n} \Big(\dots\Big(\int_0^{t_2} \mathrm{d}B_{t_1}\Big)\dots\Big)\mathrm{d}B_{t_{n-1}}\bigg)\mathrm{d}B_{t_n} \tag{20.51}$$

In particular, for all $n \geqslant 1$ and all $t \geqslant 0$, we have

$$\int_0^t \Bigl(\int_0^{t_n} \Bigl(\dots \Bigl(\int_0^{t_2} \mathrm{d}B_{t_1}\Bigr) \dots\Bigr)\mathrm{d}B_{t_{n-1}}\Bigr)\mathrm{d}B_{t_n} = \frac{t^{n/2}}{n!} h_n(B_t/\sqrt{t}) \tag{20.52}$$

where

$$h_n(x) := (-1)^n \mathrm{e}^{\frac{x^2}{2}} \frac{\mathrm{d}^n}{\mathrm{d}x^n} \mathrm{e}^{-\frac{x^2}{2}} \tag{20.53}$$

is the n-th order Hermite polynomial.

Proof. For (20.51) set $f := \lambda$ in (20.49). For the second part of the claim let $X = \mathcal{N}(0,1)$ and note that integration by parts gives us

$$E\bigl(\mathrm{e}^{\lambda X} h_n(X)\bigr) = \lambda^n E(\mathrm{e}^{\lambda X}) = \mathrm{e}^{\frac{\lambda^2}{2}} \tag{20.54}$$

The fact (which we leave to the reader) that $\{\frac{1}{\sqrt{n!}} h_n\}_{n \geqslant 0}$ form an orthonormal basis in L^2-space associated with the law of the standard normal then show

$$\mathrm{e}^{\lambda x - \frac{1}{2}\lambda^2} = 1 + \sum_{n \geqslant 1} \frac{\lambda^n}{n!} h_n(x) \tag{20.55}$$

with the sum convergent in L^2 associated with $\mathcal{N}(0,1)$. Plugging $x := B_t/\sqrt{t}$ and replacing λ by $\lambda\sqrt{t}$ then gives

$$\mathrm{e}^{\lambda B_t - \frac{1}{2}\lambda^2 t} = 1 + \sum_{n \geqslant 1} \lambda^n \frac{t^{n/2}}{n!} h_n(B_t/\sqrt{t}) \tag{20.56}$$

Identifying terms with those in (20.51) we get (20.52). □