19. BROWNIAN MARTINGALES

The results in the previous chapter showed that the standard Brownian motion plays quite a central role among all continuous local martingales. Here we examine this role in a somewhat different context.

19.1 Representing L^2-random variables.

To motivate the forthcoming derivations, recall that the Doob-Dynkin lemma states that if X and Y are random variables and X is $\sigma(Y)$ measurable, then $X = f(Y)$ for some Borel function f (between the respective spaces that we deliberately keep implicit). It follows that, if B is a standard Brownian motion and

$$\mathcal{F}_t^B := \sigma(B_s \colon s \leqslant t) \tag{19.1}$$

is its natural filtration, then any $\mathcal{F}_t^B$-measurable random variable X is a measurable function of $\{B_s \colon s \leqslant t\}$. Leaving the technicalities aside, this should also somehow mean that X is a function of the infinitesimal increments $\{\mathrm{d}B_s \colon s < t\}$. This suggests a representation in terms of a stochastic integral, which we state in the form:

Theorem 19.1 *Let B be a Brownian motion endowed with its natural filtration $\{\mathcal{F}_t^B\}_{t\geqslant 0}$. Then for each $X \in L^2$ there exists a stochastic process $Y \in \mathcal{V}_B$ such that*

$$\forall t \geqslant 0\colon \quad E(X|\mathcal{F}_t^B) = EX + \int_0^t Y_s \mathrm{d}B_s \quad \text{a.s.} \tag{19.2}$$

The process Y is determined uniquely up to the equivalence relation between processes in $\mathcal{V}_B$.

Proof. Let $X \in L^2$ and, shifting X by a constant, assume $EX = 0$. We start by the construction of Y which we first motivate as follows. Observe that $E(X|\mathcal{F}_t^B)$ is an element of the Hilbert space $L^2(\Omega, \mathcal{F}_t^B, P)$. The collection $\mathcal{I} := \{\int_0^t Y_s \mathrm{d}B_s \colon Y \in \mathcal{V}_B\}$, with a.s.-equal random variables identified as one, forms a closed linear subspace of $L^2(\Omega, \mathcal{F}_t^B, P)$. This suggests that we seek Y for which $\int_0^t Y_s \mathrm{d}B_s$ is nearest to $E(X|\mathcal{F}_t^B)$ in $\mathcal{I}$ or, alternatively, the orthogonal projection of $E(X|\mathcal{F}_t^B)$ onto $\mathcal{I}$.

Formulating the projection argument in terms of Hilbert spaces requires working with equivalence classes of random variables, so we rather proceed by mimicing the steps from Hilbert space theory for actual stochastic processes. Fix $t \geqslant 0$ and consider the quadratic functional $\varphi \colon \mathcal{V}_B \to \mathbb{R}$ defined by

$$\varphi(Y) := E\biggl(\Bigl[E(X|\mathcal{F}_t^B) - \int_0^t Y_s \mathrm{d}B_s\Bigr]^2\biggr) \tag{19.3}$$

The above suggest looking for Y achieving $c := \inf\{\varphi(Y) \colon Y \in \mathcal{V}_B\}$. Pick a minimizing sequence $\{Y^{(n)}\}_{n\in\mathbb{N}} \in \mathcal{V}_B^{\mathbb{N}}$ (i.e., one for which $\varphi(Y^{(n)}) \to c$) and observe that the quadratic nature of φ implies

$$E\biggl(\Bigl[\int_0^t Y_s^{(n)} \mathrm{d}B_s - \int_0^t Y_s^{(m)} \mathrm{d}B_s\Bigr]^2\biggr) = 2\varphi(Y^{(n)}) + 2\varphi(Y^{(m)}) - 4\varphi\Bigl(\frac{Y^{(n)} + Y^{(m)}}{2}\Bigr) \tag{19.4}$$

The right hand side is not larger than $2\varphi(Y^{(n)}) + 2\varphi(Y^{(m)}) - 4c$ and so it tends to zero as $m, n \to \infty$. The left-hand side then tends to zero in this limit as well. The Itô isometry turns that conclusion into

$$\lim_{m,n\to\infty} E\left(\int_0^t (Y_s^{(n)} - Y_s^{(m)})^2 \mathrm{d}s\right) = 0. \tag{19.5}$$

It follows that $\{Y^{(n)}\}_{n\in\mathbb{N}}$ is Cauchy in $L^2([0,t]\times\Omega)$. As we may assume that $Y_s^{(n)} = 0$ for $s \geqslant 0$, the Cauchy property holds even in $\mathcal{V}_B$.

Since $\mathcal{V}_B$ is the set of limits of all Cauchy sequences as above, it follows that there exists $Y \in \mathcal{V}_B$ such that $[\![Y^{(n)} - Y]\!]_B \to 0$. This implies $\int_0^t Y_s^{(n)} \mathrm{d}B_s \to \int_0^t Y_s \mathrm{d}B_s$ in L^2 and so $\varphi(Y^{(n)}) \to \varphi(Y)$. It follows that $\varphi(Y) = c$ and so Y is indeed a minimizer. Replacing Y by $Y + \epsilon Z$ for ϵ both positive and negative then shows

$$\forall Z \in \mathcal{V}_B\colon\ E\left(\left[E(X|\mathcal{F}_t^B) - \int_0^t Y_s \mathrm{d}B_s\right] \int_0^t Z_s \mathrm{d}B_s\right) = 0. \tag{19.6}$$

This is actually all we will need from Y in the sequel.

Fix $t \geqslant 0$ and let $0 = t_0 < \dots < t_n \leqslant t$. Define the complex-valued process

$$M_s := \exp\left\{\mathrm{i}\sum_{j=1}^n \lambda_j (B_{t_j\wedge s} - B_{t_{j-1}\wedge s}) + \frac{1}{2}\sum_{j=1}^n \lambda_j^2 (t_j \wedge s - t_{j-1} \wedge s)\right\} \tag{19.7}$$

which is a martingale because the exponent is of the form $\widetilde{M}_s - \frac{1}{2}\langle \widetilde{M}\rangle_s$ for $\widetilde{M}$ a continuous martingale. The Itô formula shows that $\mathrm{d}M_s = Z_s \mathrm{d}B_s$ for

$$Z_s := \Bigl(\mathrm{i}\sum_{j=1}^n \lambda_j 1_{(t_{j-1},t_j]}(s)\Bigr) M_s \tag{19.8}$$

and so

$$\int_0^t Z_s \mathrm{d}B_s = M_t - M_0 = M_t - 1 \tag{19.9}$$

Abbreviating

$$V_t := E(X|\mathcal{F}_t^B) - \int_0^t Y_s \mathrm{d}B_s \tag{19.10}$$

from (19.6) we thus get $E(V_t M_t) = E(V_t) = E(X) = 0$. Introducing $\lambda_j' := \lambda_j - \lambda_{j+1}$ for $j = 1, \dots, n-1$ and $\lambda_n' := \lambda_n$, this readily translates into

$$E\left(V_t \exp\Bigl\{\mathrm{i}\sum_{j=1}^n \lambda_j' B_{t_j}\Bigr\}\right) = 0 \tag{19.11}$$

Since this holds for all $\lambda_1', \dots, \lambda_n'$, integrating this against the Fourier transform of a function $f \in L^2(\mathbb{R}^n)$ shows

$$E\bigl(V_t f(B_{t_1}, \dots, B_{t_n})\bigr) = 0. \tag{19.12}$$

Specializing to indicators of bounded Borel sets is sufficient to conclude

$$E\bigl(V_t \,\big|\, \sigma(B_{t_1}, \dots, B_{t_n})\bigr) = 0 \quad \text{a.s.} \tag{19.13}$$

As the ordering of the t_i's no longer matters, this is true for all natural $n \geqslant 1$ and all $t_1, \dots, t_n \in [0,t]$.

Let $D_n := \{k2^{-n}t \colon k = 1, \dots, 2^n\}$ and set $D := \bigcup_{n\geqslant 0} D_n$. Note that $D_n \subseteq D_{n+1}$ gives $\sigma(B_u \colon u \in D_n) \subseteq \sigma(B_u \colon u \in D_{n+1})$ and $\sigma(\bigcup_{n\geqslant 1} \sigma(B_u \colon u \in D_n)) = \sigma(B_u \colon u \in D)$. We also readily check that $\sigma(B_u \colon u \in D) = \mathcal{F}_t^B$ by continuity of B. The Lévy Forward Theorem along with (19.13) then shows

$$E(V_t|\mathcal{F}_t^B) = 0 \quad \text{a.s.} \tag{19.14}$$

which by the fact that V_t is $\mathcal{F}_t^B$-measurable implies $V_t = 0$ a.s.

To complete the proof let us write $Y^{(t)}$ for the process defined for t as above. Observe that, for $u \leqslant t$, we then have

$$\begin{aligned}\int_0^u Y_s^{(t)} \mathrm{d}B_s &= E\Bigl(\int_0^t Y_s^{(t)} \mathrm{d}B_s \Bigm| \mathcal{F}_u^B\Bigr) \\ &= E\bigl(E(X|\mathcal{F}_t^B) - EX \bigm| \mathcal{F}_u^B\bigr) \\ &= E(X|\mathcal{F}_u^B) - EX = \int_0^u Y_s^{(u)} \mathrm{d}B_s \quad \text{a.s.}\end{aligned} \tag{19.15}$$

Itô isometry shows that $\{s \leqslant u \colon Y_s^{(t)} \neq Y_s^{(u)}\}$ has zero Lebesgue measure a.s. Setting

$$Y_s := \sum_{t\geqslant 1} Y_s^{(t)} 1_{(t-1,t]}(s) \tag{19.16}$$

then gives us $Y \in \mathcal{V}_B$ such that (19.2) holds. □

Note that we can replace $E(X|\mathcal{F}_t^B)$ by X if we know that X is $\mathcal{F}_t^B$-measurable. The requirement $X \in L^2$ was quite useful in the proof but the Doob-Dynkin lemma seems to work without any assumption on integrability. A representation under just a.s. finiteness was established by Dudley in 1977; see Theorem 4.20 in Karatzas and Shreve. Note, however, that without some integrability, the uniqueness of the representation is lost. This is because, for each $t > 0$, one can find $Y \in \mathcal{V}_B^{\text{loc}}$ such that

$$\int_0^t Y_s^2 \mathrm{d}s > 0 \quad \text{yet} \quad \int_0^t Y_s \mathrm{d}B_s = 0 \quad \text{a.s.} \tag{19.17}$$

We leave a proof of this fact to a homework exercise.

19.2 Extension to Brownian L^2-martingales.

Suppose now that a process M is defined on the same space as the standard Brownian motion B. Consider the *augmented* filtration $\{\widetilde{\mathcal{F}}_t^B\}_{t\geqslant 0}$ defined by

$$\widetilde{\mathcal{F}}_t^B := \sigma(\mathcal{F}_t^B \cup \mathcal{N}), \tag{19.18}$$

for $\mathcal{F}_t^B$ as in (19.1) and $\mathcal{N}$ the collection of all P-null sets. If M is adapted and satisfies the defining relation of a martingale relative to $\{\widetilde{\mathcal{F}}_t^B\}_{t\geqslant 0}$, then we refer to it as a *Brownian martingale*. The above theorem allows us to characterize all Brownian martingales as stochastic integrals:

Theorem 19.2 *Let $(\Omega, \mathcal{F}, P)$ be a probability space supporting a standard Brownian motion B and a stochastic process M that is an L^2-martingale with respect to a filtration $\{\mathcal{F}_t\}_{t\geqslant 0}$ satisfying $\forall t \geqslant 0\colon \mathcal{F}^B_t \subseteq \mathcal{F}_t \subseteq \widetilde{\mathcal{F}}^B_t$. Then there exists $Y \in \mathcal{V}_B$, adapted to $\{\mathcal{F}^B_t\}_{t\geqslant 0}$, such that*

$$\forall t \geqslant 0\colon\; M_t = M_0 + \int_0^t Y_s \mathrm{d}B_s \quad \text{a.s.} \tag{19.19}$$

The process Y is determined uniquely up to the equivalence relation between processes in $\mathcal{V}_B$.

Proof. Given any natural $n \geqslant 1$, note that

$$E(M_n|\mathcal{F}^B_t) = E(M_n|\mathcal{F}_t) = M_{t\wedge n} \quad \text{a.s.} \tag{19.20}$$

by the martingale property and the fact that each $A \in \mathcal{F}^B_t$ differs from some $A' \in \widetilde{\mathcal{F}}^B_t$ by a null set. Similarly,

$$M_0 = E(M_0) \quad \text{a.s.} \tag{19.21}$$

by the fact that P is trivial on $\mathcal{F}_0$. With this in mind, Theorem 19.1 yields existence of $Y^{(n)} \in \mathcal{V}_B$, adapted to $\{\mathcal{F}^B_t\}_{t\geqslant 0}$, such that

$$\forall t \geqslant 0\colon \quad M_{t\wedge n} = M_0 + \int_0^t Y^{(n)}_s \mathrm{d}B_s \quad \text{a.s.} \tag{19.22}$$

Taking this for n replaced by $n+1$ at $t = n$, we get

$$\int_0^n Y^{(n+1)}_s \mathrm{d}B_s = \int_0^n Y^{(n)}_s \mathrm{d}B_s \quad \text{a.s.} \tag{19.23}$$

In light of square integrability, the Itô isometry then gives

$$\int_0^n \bigl(Y^{(n+1)}_s - Y^{(n)}_s\bigr)^2 \mathrm{d}s = 0 \quad \text{a.s.} \tag{19.24}$$

meaning that the two processes are equivalent as elements of $L^2([0,n]\times\Omega)$. Setting

$$Y_t := \sum_{n\geqslant 1} Y^{(n)}_t 1_{(n-1,n]}(t) \tag{19.25}$$

this means that we can replace $Y^{(n)}$ by $\{Y_s 1_{[0,n]}(s)\colon s \geqslant 0\}$ in (19.22) without affecting the a.s. equality. The indicator is removed by taking $n \to \infty$ with the help of Itô isometry and Dominated convergence, which then yields the desired claim. □

Note that we have note made any assumption of continuity of the martingale which is mainly because the proof is done for each time separately. Notwithstanding, the result gives continuity of M as a corollary:

Corollary 19.3 *Let M be an L^2-martingale with respect to the filtration $\{\widetilde{\mathcal{F}}^B_t\}_{t\geqslant 0}$, where B is a standard Brownian motion and $\widetilde{\mathcal{F}}^B_t$ is as in* (19.18)*. Then M admits a continuous version.*

Proof. By Theorem 19.2, $\{M_0 + \int_0^t Y_s \mathrm{d}B_s\colon t \geqslant 0\}$ is a version of M. Since, for our choice of the filtration, the stochastic integrals admit a continuous version, so does M. □

As a consequence of the Martingale Convergence Theorem, a.e. sample of a martingale has left and right limits along rationals at all (positive) times. For filtrations that are one-sided continuous, the corresponding one-sided limit is a version of the martingale. However, this does not imply that for filtrations that are continuous (as is $\widetilde{\mathcal{F}}_t^B$ above) the martingale is continuous. (Indeed, this fails for the Poisson process endowed with its augmented filtration.) So Corollary 19.3 does say something more than general arguments would imply.

Further reading: Karatzas-Shreve, Section 3.4D