16. Integrability and localization

Continuing our discussion of the Itô integral with respect to continuous martingales, next we focus on criteria for integrability and extension of the integral to local martingales and locally integrable processes.

16.1 Which processes can we integrate?.

The stochastic integral would hardly be very useful if we cannot supply an independent characterization of the class of processes that can be integrated. The answer to this turns out to be somewhat more subtle than for the Itô integral with respect to standard Brownian motion. To see the reason, note that two processes $Y, \widetilde{Y} \in \mathcal{V}_M$ are in the same equivalence class (as elements of $\mathcal{V}_M$) if

$$\forall t \geqslant 0\colon \quad E\biggl(\int_0^t (Y_s - \widetilde{Y}_s)^2 \mathrm{d}\langle M\rangle_s\biggr) = 0 \tag{16.1}$$

Here, if $t \mapsto \langle M\rangle_t$ is absolutely continuous a.s., Fubini-Tonellis tells us that this happens if and only if $\widetilde{Y}$ is a version of Y in the sense that $P(Y_t = \widetilde{Y}_t) = 1$ for all $t \geqslant 0$. However, this changes radically if $t \mapsto \langle M\rangle_t$ is not absolutely continuous a.s. because then the set $\{t \geqslant 0\colon Y_t \neq \widetilde{Y}_t\}$ may have fail to have vanishing Lebesgue measure while (16.1) still holds, and *vice versa*. We thus have to treat the two cases separately.

The absolutely continuous case reduces pretty much to familiar arguments:

Proposition 16.1 *For all $M \in \mathscr{M}_2^{\text{cont}}$,*

$$t \mapsto \langle M\rangle_t \text{ absolutely continuous a.s.} \quad \Rightarrow \quad \overline{\mathcal{V}_0}^{[\![\cdot]\!]_M} = \mathcal{V}_M \tag{16.2}$$

Proof. Let $Y \in \mathcal{V}_M$ and suppose first that Y is bounded by some $K > 0$. Theorem 10.5 gives existence of a sequence $\{Y^{(n)}\}_{n\in\mathbb{N}} \in \mathcal{V}_0$ such that

$$\forall t \geqslant 0\colon \; E\biggl(\int_0^t (Y_s - Y_s^{(n)})^2 \mathrm{d}s\biggr) \underset{n\to\infty}{\longrightarrow} 0. \tag{16.3}$$

We may also assume that the processes $\{Y^{(n)}\}_{n\in\mathbb{N}}$ are bounded by the same constant as Y. Resorting to a subsequence if necessary, a Borel-Cantelli argument in turn permits us to assume that

$$\Bigl\{t \geqslant 0\colon \limsup_{n\to\infty} |Y_t - Y_t^{(n)}| > 0\Bigr\} \tag{16.4}$$

has vanishing Lebesgue measure P-a.s.

The assumed absolute continuity of $t \mapsto \langle M\rangle_t(\omega)$ now implies the existence of a (random) Radon-Nikodym derivative $F_M := \frac{\mathrm{d}\langle M\rangle_t}{\mathrm{d}t}$ which is a locally Lebesgue integrable function $F_M\colon \mathbb{R}_+ \to \mathbb{R}_+$ such that

$$\int_0^t (Y_s - Y_s^{(n)})^2 \mathrm{d}\langle M\rangle_s = \int_0^t (Y_s - Y_s^{(n)})^2 F_M(s) \mathrm{d}s \tag{16.5}$$

holds for each $t \geqslant 0$. In light of the Lebesgue-null property of the set (16.4) and the boundedness of $Y - Y^{(n)}$, the integral on the right converges to zero P-a.s. by the Dominated Convergence Theorem. But the integral on the left is bounded by $4K^2\langle M\rangle_t$, which is in L^1. The Dominated Convergence Theorem then shows

$$\forall t \geqslant 0\colon\ E\biggl(\int_0^t (Y_s - Y_s^{(n)})^2 \mathrm{d}\langle M\rangle_s\biggr) \underset{n\to\infty}{\longrightarrow} 0 \tag{16.6}$$

proving that $[\![Y - Y^{(n)}]\!]_M \to 0$ and thus $Y \in \overline{\mathcal{V}_0}^{[\![\cdot]\!]_M}$ whenever $Y \in \mathcal{V}_M$ is bounded.

If $Y \in \mathcal{V}_M$ is unbounded, we set $Y_s^{(n)} = Y_s 1_{\{|Y_s|\leqslant n\}}$ and observe that $Y^{(n)} \in \mathcal{V}_M$ is bounded and so $Y^{(n)} \in \overline{\mathcal{V}_0}^{[\![\cdot]\!]_M}$ by the previous argument. But then

$$E\biggl(\int_0^t (Y_s - Y_s^{(n)})^2 \mathrm{d}\langle M\rangle_s\biggr) = E\biggl(\int_0^t Y_s^2 1_{\{|Y_s|>n\}} \mathrm{d}\langle M\rangle_s\biggr) \tag{16.7}$$

and the right-hand side tends to zero as $n \to \infty$ by the Dominated Convergence Theorem and the fact that $Y \in \mathcal{V}_M$. Hence $Y \in \overline{\mathcal{V}_0}^{[\![\cdot]\!]_M}$ in this case as well. □

16.2 Progressively measurable processes.

For the case when $t \mapsto \langle M\rangle_t$ is not absolutely continuous, we will have to impose the following stronger measurability condition:

Definition 16.2 *Given a filtration $\{\mathcal{F}_t\}_{t\geqslant 0}$ on a measure space $(\Omega, \mathcal{F})$, a stochastic process $\{X_t\colon t \geqslant 0\}$ is* progressively measurable *if for all $t \geqslant 0$,*

$$\bigl\{(\omega, s) \in \Omega \times [0, t]\colon X_s(\omega) \in A\bigr\} \in \mathcal{F}_t \otimes \mathcal{B}([0, t]) \tag{16.8}$$

holds for each $A \in \mathcal{B}(\mathbb{R})$.

Intersecting the set in (16.8) by $\Omega \times \{t\}$, which lies in $\mathcal{F}_t \otimes \mathcal{B}([0, t])$ thanks to singletons being Borel measurable, we find out that a progressively measurable process is automatically adapted and, by the fact that $\bigcup_{t\geqslant 0} \mathcal{F}_t \subseteq \mathcal{F}$, it is also jointly measurable. Unfortunately, the converse does not hold in general because of the lack of any relation (besides inclusion) between $\mathcal{F}_t$ and $\mathcal{F}$.

As is readily checked, an adapted process that has left-continuous paths is automatically progressively measurable. The class of progressively measurable processes thus includes the so called *previsible* (a.k.a. *predictable*) processes, which is the smallest class of processes closed under pointwise limits and containing all adapted left-continuous processes. (Previsible processes play an important role in integration theory with respect to discontinuous martingales.)

The key additional structure supplied by progressive measurability becomes quite apparent in the proof of the following lemma:

Lemma 16.3 *Let T be a stopping time for the filtration $\{\mathcal{F}_t\}_{t\geqslant 0}$ on $(\Omega, \mathcal{F})$. Then T is measurable with respect to the σ-algebra*

$$\mathcal{F}_T := \bigl\{A \in \mathcal{F}\colon (\forall t \geqslant 0\colon A \cap \{T \leqslant t\} \in \mathcal{F}_t)\bigr\} \tag{16.9}$$

If $\{X_t\colon t\geqslant 0\}$ is progressively measurable and T is finite, then also X_T is $\mathcal{F}_T$-measurable.

Proof. Let $u,t\geqslant 0$. Then $\{T\leqslant u\}\cap\{T\leqslant t\}=\{T\leqslant t\wedge u\}\in\mathcal{F}_{t\wedge u}\subseteq\mathcal{F}_t$. Hence $\{T\leqslant u\}\in\mathcal{F}_T$ for each $u\geqslant 0$ which by the fact that sets of the form $\{T\leqslant u\}$ generate $\sigma(T)$ shows that T is indeed $\mathcal{F}_T$-measurable.

Next, given $t\geqslant 0$, progressive measurability means that $(\omega,s)\mapsto X_s(\omega)$, as a map $\Omega\times[0,t]\to\mathbb{R}$, is $\mathcal{F}_t\otimes\mathcal{B}([0,t])/\mathcal{B}(\mathbb{R})$-measurable. The fact that $T\wedge t$ is a stopping time in turn gives that $\omega\mapsto(\omega,T(\omega)\wedge t)$ is $\mathcal{F}_t/\mathcal{F}_t\otimes\mathcal{B}([0,t])$-measurable. The composition of the two maps $\omega\mapsto X_{T(\omega)\wedge t}(\omega)$ is thus $\mathcal{F}_t/\mathcal{B}(\mathbb{R})$-measurable. This means that, for all $B\in\mathcal{B}(\mathbb{R})$, we have $\{X_{T\wedge t}\in B\}\in\mathcal{F}_t$. But then also

$$\{X_T\in B\}\cap\{T\leqslant t\}=\bigl\{(X_{T\wedge t},T\wedge t)\in B\times[0,t]\bigr\}\cap\{T\leqslant t\}\in\mathcal{F}_t \tag{16.10}$$

showing that $\{X_T\in B\}\in\mathcal{F}_T$, as desired. □

We note that, for X as well as the filtration are right-continuous, one can prove that X_T is $\mathcal{F}_T$-measurable directly by approximations of the stopping time. Progressive measurability is thus a type of regularity condition that replaces continuity assumptions when these cannot be taken for granted.

Returning to the question of our main interest, we now state:

Proposition 16.4 *For all $M\in\mathscr{M}_2^{\text{cont}}$ and all $Y\in\mathcal{V}_M$,*

$$Y\text{ progressively measurable}\quad\Rightarrow\quad Y\in\overline{\mathcal{V}_0}^{[\![\cdot]\!]_M} \tag{16.11}$$

Proof. The proof proceeds by a random time change that, effectively, reduces it again to the case (16.3) above. Progressive measurability ensures that the time-changed process is adapted to the time-changed filtration; see Lemma 16.3.

A natural process to base the time change on is $t\mapsto\langle M\rangle_t$. Unfortunately, this process may not be a strictly increasing (and thus one-to-one) function and so we instead work with $t\mapsto\langle M\rangle_t+t$. The inverse is supplied by the stopping times

$$T(u):=\inf\bigl\{t\geqslant 0\colon\langle M\rangle_t+t\geqslant u\bigr\} \tag{16.12}$$

for which we get

$$\forall u\geqslant 0\colon\;\; T(u)+\langle M\rangle_{T(u)}=u. \tag{16.13}$$

The process $t\mapsto T(u)$ is continuous and strictly increasing with $T(0)=0$.

Let now $Y\in\mathcal{V}_M$ be as in the statement. The argument in the previous proof permits us to assume that Y is bounded, say, $|Y|_t\leqslant K$ for all $t\geqslant 0$. The progressive measurability ensures that $\{Y_{T(u)}\colon u\geqslant 0\}$ is adapted to the filtration $\{\mathcal{F}_{T(u)}\}_{u\geqslant 0}$ and the joint measurability of $\{Y_t\colon t\geqslant 0\}$ and $\{T(u)\colon u\geqslant 0\}$ (implied, in the latter case, by continuity) shows that $\{Y_{T(u)}\colon u\geqslant 0\}$ is jointly measurable. Since $E[\int_0^t Y_s^2\mathrm{d}s]\leqslant Kt$ for each $t\geqslant 0$, the aforementioned lemma from 275D yields existence of processes $\{Y^{(n)}\}_{n\in\mathbb{N}}$ of the form

$$Y_u^{(n)}=Z_0^{(n)}1_{\{0\}}(u)+\sum_{i=1}^{m_n}Z_i^{(n)}1_{(t_{i-1}^n,t_i^n]}(u) \tag{16.14}$$

with $m \in \mathbb{N}$, $0 = t_0^n < \cdots < t_{m_n}$ such that

$$\forall n \in \mathbb{N}\, \forall i = 0, \dots, m_n\colon \ Z_i^{(n)} \text{ is } \mathcal{F}_{T(t_{i-1}^n \vee 0)}\text{-measurable} \tag{16.15}$$

for which

$$\forall t \geqslant 0\colon \ E\biggl(\int_0^t (Y_{T(u)} - Y_u^{(n)})^2 \mathrm{d}u\biggr) \underset{n\to\infty}{\longrightarrow} 0. \tag{16.16}$$

Abbreviating

$$\widetilde{Y}_s^{(n)} := Y^{(n)}_{\langle M\rangle_s + s} \tag{16.17}$$

which is jointly measurable thanks to the fact that $s \mapsto \langle M\rangle_s$ is jointly measurable, the substitution $u := \langle M\rangle_s + s$ inside the integral then shows, for each $t \geqslant 0$, that

$$\begin{aligned} E\biggl(\int_0^t (Y_s - \widetilde{Y}_s^{(n)})^2 \mathrm{d}\langle M\rangle_s\biggr) &\leqslant E\biggl(\int_0^t (Y_s - Y^{(n)}_{\langle M\rangle_s + s})^2 (\mathrm{d}\langle M\rangle_s + \mathrm{d}s)\biggr) \\ &= E\biggl(\int_0^{t+\langle M\rangle_t} (Y_{T(u)} - Y_u^{(n)})^2 \mathrm{d}u\biggr) \\ &\leqslant E\biggl(\int_0^{t+v} (Y_{T(u)} - Y_u^{(n)})^2 \mathrm{d}u\biggr) + 4K^2 E\bigl((t + \langle M_t\rangle) 1_{\{\langle M\rangle_t \geqslant v\}}\bigr), \end{aligned} \tag{16.18}$$

where we used that $(Y_{T(u)} - Y_u^{(n)})^2 \leqslant 4K^2$. The first term on the right now tends to zero as $n \to \infty$ by (16.16) while the second term tends to zero as $v \to \infty$ using the Dominated Convergence Theorem. We conclude that $[\![Y - \widetilde{Y}^{(n)}]\!]_M \to 0$.

To get the claim it thus suffices to show that $\widetilde{Y}^{(n)} \in \overline{\mathcal{V}_0}^{[\![\cdot]\!]_M}$ for each $n \geqslant 0$. For this we observe that the continuity and strict monotonicity of $t \mapsto T(t)$ gives

$$\widetilde{Y}_u^{(n)} = Z_0^{(n)} 1_{\{0\}}(u) + \sum_{i=1}^{m_n} Z_i^{(n)} 1_{(T(t_{i-1}^n), T(t_i^n)]}(u). \tag{16.19}$$

Defining $T_k(t) := 2^{-k}\lceil 2^k T(t)\rceil$, we have $T_k(t) \downarrow T(t)$ and so $1_{(T_k(s), T_k(u)]} \to 1_{(T(s), T(u)]}$ pointwise as $k \to \infty$ for all $s < u$. It follows that, as $k \to \infty$,

$$\widetilde{Y}_u^{(n,k)} := Z_0^{(n)} 1_{\{0\}}(u) + \sum_{i=1}^{m_n} Z_i^{(n)} 1_{(T_k(t_{i-1}^n), T_k(t_i^n)]}(u) \tag{16.20}$$

converges to $\widetilde{Y}_u^{(n)}$ for each $u \geqslant 0$. The limit $[\![\widetilde{Y}^{(n,k)} - \widetilde{Y}^{(n)}]\!]_M \to 0$ as $k \to \infty$ then takes place by the Dominated Convergence Theorem.

We will now show that $Y^{(n,k)} \in \mathcal{V}_0$. Indeed, thanks to the dyadic approximation of the stopping times, $(T_k(t_{i-1}^n), T_k(t_i^n)]$ is the union of intervals $(2^{-k}(j-1), 2^{-k}j]$ for j such that $T_k(t_{i-1}^n) < 2^{-k}j \leqslant T_k(t_i^n)$. Noting that $T_k(t_i^n) \leqslant t_{m_n}^n$ and setting $r_{n,k} := \lceil 2^k t_{m_n}^n\rceil$, we can thus rewrite the right-hand side of (16.20) as

$$\widetilde{Y}_u^{(n,k)} = Z_0^{(n)} 1_{\{0\}}(u) + \sum_{i=1}^{m_n} \sum_{j=1}^{r_{n,k}} \bigl(Z_i^{(n)} 1_{\{T_k(t_{i-1}^n) < 2^{-k}j \leqslant T_k(t_i^n)\}}\bigr) 1_{(2^{-k}(j-1), 2^{-k}j]}(u). \tag{16.21}$$

Now observe that $\{T_k(t^n_{i-1}) < 2^{-k}j \leqslant T_k(t^n_i)\} \in \mathcal{F}_{2^{-k}(j-1)}$ and that $Z^{(n)}_i$ is $\mathcal{F}_{t^n_{i-1}}$-measurable and thus also $\mathcal{F}_{2^k\lceil t^n_{i-1}\rceil}$-measurable. Hence we get that

$$Z^{(n)}_i 1_{\{T_k(t^n_{i-1})<2^{-k}j\leqslant T_k(t^n_i)\}} \text{ is } \mathcal{F}_{2^{-k}(j-1)}\text{-measurable} \tag{16.22}$$

for each $i = 1,\dots,m_n$ and each $j = 1,\dots,r_{n,k}$. This implies $\widetilde{Y}^{(n,k)} \in \mathcal{V}_0$ for each $n,k \geqslant 0$ and, by above reasoning, $\widetilde{Y}^{(n)} \in \overline{\mathcal{V}_0}^{[\![\cdot]\!]_M}$ for each $n \geqslant 0$. □

That we need to ask more from Y when we ask less from $\langle M\rangle$ is a well known fact in Stieltjes integration theory where this is often used to trade regularity of the integrand against the regularity of the integrating function. Still, one is left to wonder what is $\overline{\mathcal{V}_0}^{[\![\cdot]\!]_M}$ when $t \mapsto \langle M\rangle_t$ is not absolutely continuous. A criterion is offered in:

Lemma 16.5 *Let $M \in \mathscr{M}^{\text{cont}}_2$ and $Y \in \mathcal{V}_M$. Then the conditions (1) and (2) in*

(1) $Y \in \overline{\mathcal{V}_0}^{[\![\cdot]\!]_M}$
(2) $\exists\widetilde{Y} \in \mathcal{V}_M$ *progressively measurable such that*

$$\int_0^\infty 1_{\{Y_t\neq\widetilde{Y}_t\}}\mathrm{d}\langle M\rangle_t = 0 \quad \text{a.s.} \tag{16.23}$$

are equivalent.

Proof. Let $Y, \widetilde{Y} \in \mathcal{V}_M$ be such that (16.23) holds. If $\widetilde{Y}$ is progressively measurable, then Proposition 16.4 gives $\widetilde{Y} \in \overline{\mathcal{V}_0}^{[\![\cdot]\!]_M}$ and so there is $\{Y^{(n)}\}_{n\in\mathbb{N}} \in \mathcal{V}_0^{\mathbb{N}}$ such that $[\![\widetilde{Y} - Y^{(n)}]\!] \to 0$. But, with (16.23) in force, Y and $\widetilde{Y}$ are indistinguishable as far as integrals with respect to $\mathrm{d}\langle M\rangle$ are concerned, and we thus have $[\![Y - Y^{(n)}]\!] \to 0$, proving (2) ⇒ (1).

For (1) ⇒ (2), let $Y \in \overline{\mathcal{V}_0}^{[\![\cdot]\!]_M}$. Then there exists $\{Y^{(n)}\}_{n\in\mathbb{N}} \in \mathcal{V}_0^{\mathbb{N}}$ with $[\![Y - Y^{(n)}]\!] \to 0$. Reducing to a subsequence, we may assume that the convergence is almost everywhere in the sense that

$$\int_0^\infty 1_{\{\limsup_{n\to\infty}|Y^{(n)}_t - Y_t|>0\}}\mathrm{d}\langle M\rangle_t = 0, \quad \text{a.s.} \tag{16.24}$$

Set

$$\widetilde{Y}_t := \limsup_{n\to\infty} Y^{(n)}_t \tag{16.25}$$

whenever the right-hand side is finite and put $\widetilde{Y}_t := 0$ otherwise. At $t \geqslant 0$ where $\limsup_{n\to\infty}|Y^{(n)}_t - Y_t| = 0$ we then have $\widetilde{Y}_t = Y_t$ (we are assuming that Y is finite everywhere) and so (16.24) implies (16.23). As simple processes are trivially progressively measurable, so is $\limsup_{n\to\infty} Y^{(n)}_t$ and thus also $\widetilde{Y}$. □

Examples of processes that are measurable, adapted, but not progressively measurable exist although they are quite contrived for the fact that they invariably capitalize on a huge difference between $\mathcal{F}$ (which determines joint measurability) and $\mathcal{F}_t$ (which determines progressive measurability). Still, assuming progressive measurability from the outset is not a considerable loss because of the following result:

Theorem 16.6 *Given a probability space $(\Omega, \mathcal{F}, P)$ and a jointly measurable process Y, for each filtration $\{\mathcal{F}_t\}_{t\geqslant 0}$ there exists a progressively measurable process $\widetilde{Y}$ that is a version of Y in the sense that $\forall t \geqslant 0\colon\ P(Y_t \neq \widetilde{Y}_t) = 0$.*

Proof. This can found in several advanced texts albeit, apparently, with difficult proofs. A simple proof has appeared recently in M. Onderját and J. Seilder's paper "On existence of progressively measurable modifictions" published in Electronic Communications in Probability, vol. 18, no. 20, year 2013, pages 1-6. The key step of that proof is to show that, given any $B \in \mathcal{B}(\mathbb{R}_+) \otimes \mathcal{F}$, the process $\{E(1_B(t,\cdot)|\mathcal{F}_t)\colon t \geqslant 0\}$ admits a progressively measurable version; the rest are approximation arguments. The proof works in great generality; namely, for Y taking values in any Polish space. □

Since (by Tonnelli), any two versions Y and $\widetilde{Y}$ of the same process necessarily agree away from a Lebesgue null-set of times a.s., the criterion (16.23) holds for any $M \in \mathscr{M}_2^{\text{cont}}$ with absolutely continuous $\langle M\rangle$, thus reproducing the conclusion of Proposition 16.1 from Proposition 16.4 and Theorem 16.6.

16.3 Localized Itô integral.

As for the integrals with respect to Brownian motion, the restriction of the above stochastic integral to square integrable integrands and integrators is too restrictive. The extension proceeds via familiar localization arguments. Given $M \in \mathscr{M}_{\text{loc}}^{\text{cont}}$ with $t \mapsto \langle M\rangle_t$ absolutely continuous a.s., set

$$\mathcal{V}_M^{\text{loc}} := \left\{Y\colon \text{measurable} \wedge \text{adapted} \wedge \forall t\geqslant 0\colon \int_0^t Y_s^2 \mathrm{d}\langle M\rangle_s < \infty \text{ a.s.}\right\}, \tag{16.26}$$

while for $t \mapsto \langle M\rangle_t$ that are not absolutely continuous a.s. we will also require that Y is progressively measurable (which subsumes measurability and adaptedness). In both cases we refer to $Y \in \mathcal{V}_M^{\text{loc}}$ as the processes that are *locally integrable* with respect to M. We then have:

Theorem 16.7 *Let $M \in \mathscr{M}_{\text{loc}}^{\text{cont}}$ and $Y \in \mathcal{V}_M^{\text{loc}}$. For each $K \geqslant 0$ set*

$$\tau_K := \inf\left\{t \geqslant 0\colon \langle M\rangle_t \geqslant K \vee \int_0^t Y_s^2 \mathrm{d}\langle M\rangle_t \geqslant t\right\} \tag{16.27}$$

and denote $M_t^{(K)} := M_{t\wedge\tau_K}$. Then $M^{(K)} \in \mathscr{M}_2^{\text{cont}}$ and $Y \in \mathcal{V}_{M^{(K)}}$. Moreover,

$$\forall L \geqslant K > 0\ \forall t \geqslant 0\colon\quad \int_0^t Y_s\,\mathrm{d}M_s^{(L)} = \int_0^t Y_s\,\mathrm{d}M_s^{(K)} \quad \text{a.s. on } \{\tau_K > t\} \tag{16.28}$$

and so

$$\forall t \geqslant 0\colon\quad \int_0^t Y_s\,\mathrm{d}M_s := \lim_{K\to\infty}\int_0^t Y_s\,\mathrm{d}M_s^{(K)} \text{ exists a.s.} \tag{16.29}$$

Moreover, assuming $\mathcal{F}_0$ contains all P-null sets, the process $\{\int_0^t Y_s \mathrm{d}M_s \colon t \geqslant 0\}$ admits a continuous version $I(Y) := \{I_t(Y) \colon t \geqslant 0\}$ which is a continuous local martingale with

$$\forall t \geqslant 0\colon \quad \big\langle I(Y)\big\rangle_t = \int_0^t Y_s^2 \mathrm{d}\langle M\rangle_s \quad \text{a.s.} \tag{16.30}$$

Finally, for any stopping time T and any $t \geqslant 0$,

$$\int_0^{T\wedge t} Y_s \mathrm{d}M_s := I_{T\wedge t}(Y) = \int_0^t Y_s 1_{\{T>s\}} \mathrm{d}M_s = \int_0^t Y_s \mathrm{d}M_{T\wedge s} \quad \text{a.s.} \tag{16.31}$$

where the integral on the right is in the sense (16.29).

Since the proof is almost exactly the same as for the Brownian case, we proceed only by some remarks. Note that in this case we need to truncate both the quadratic variation of the integral to be defined and that of the underlying martingale. This was not needed when M was a standard Brownian motion because then $\langle M\rangle$ was explicit and deterministic. Also note that for Brownian motion we truncated the integrals slightly differently; namely, by writing $\int_0^t Y_s 1_{\{\tau_K>s\}} \,\mathrm{d}M_s$ instead of $\int_0^t Y_s \,\mathrm{d}M_s^{(K)}$. The reason is that the former integral is still only in the localized sense while the latter is a proper L^2-Itô integral. That these are the same follows from (16.31).

The last clause (16.31) in the above theorem shows that

$$\forall t \geqslant 0\colon \quad \int_0^t Y_s \,\mathrm{d}M_{T_n\wedge s} \underset{n\to\infty}{\longrightarrow} \int_0^t Y_s \,\mathrm{d}M_s \quad \text{a.s.} \tag{16.32}$$

whenever $\{T_n\}_{n\geqslant 1}$ is a sequence of stopping times such that $M_t^{(n)} := M_{T_n\wedge t}$ obeys

$$\forall n \geqslant 1\colon \quad M^{(n)} \in \mathscr{M}_2^{\text{cont}} \ \wedge \ Y \in \mathcal{V}_M^{\text{loc}} \cap \mathcal{V}_{M^{(n)}} \tag{16.33}$$

and we have $T_n \to \infty$ a.s. This means that we can localize by any sequence of stopping times that makes both the integrator and the integrand square integrable with respect to the underlying probability measure. As a corollary we obtain the following statements of linearity with respect to the integrand and integrator:

Lemma 16.8 *For all $M \in \mathscr{M}_{\text{loc}}^{\text{cont}}$, all $Y, \widetilde{Y} \in \mathcal{V}_M^{\text{loc}}$ and all $\alpha, \beta \in \mathbb{R}$ we have $\alpha Y + \beta\widetilde{Y} \in \mathcal{V}_M^{\text{loc}}$ and, for all $t \geqslant 0$,*

$$\int_0^t (\alpha Y + \beta\widetilde{Y})_s \mathrm{d}M_s = \alpha \int_0^t Y_s \mathrm{d}M_s + \beta \int_0^t \widetilde{Y}_s \mathrm{d}M_s \quad \text{a.s.} \tag{16.34}$$

Lemma 16.9 *For all $M, \widetilde{M} \in \mathscr{M}_{\text{loc}}^{\text{cont}}$ and all $\alpha, \beta \in \mathbb{R}$ we have $\alpha M + \widetilde{M} \in \mathcal{V}^{\text{loc}}$ and, for all $Y \in \mathcal{V}_M^{\text{loc}} \cap \mathcal{V}_{\widetilde{M}}^{\text{loc}}$, also $Y \in \mathcal{V}_{\alpha M+\beta\widetilde{M}}^{\text{loc}}$. Moreover, for all $t \geqslant 0$,*

$$\int_0^t Y_s \mathrm{d}(\alpha M + \beta\widetilde{M})_s = \alpha \int_0^t Y_s \mathrm{d}M_s + \beta \int_0^t Y_s \mathrm{d}\widetilde{M}_s \quad \text{a.s.} \tag{16.35}$$

These are verified readily for simple integrands and, taking L^2-limits, for square integrable processes. The extension to locally integrable processes invokes localization along sequences of local times that truncate all required objects simultaneously.

Similarly as for the Itô integral with respect to standard Brownian motion, also here the localized version can be ultimately obtained directly by approximating the processes and martingales directly in probability.

Lemma 16.10 *Let $M \in \mathscr{M}_{\mathrm{loc}}^{\mathrm{cont}}$ and suppose that $Y \in \mathcal{V}_M^{\mathrm{loc}}$ and $\{Y^{(n)}\}_{n\in\mathbb{N}} \in (\mathcal{V}_M^{\mathrm{loc}})^{\mathbb{N}}$ are such that, for some $t \geqslant 0$,*

$$\int_0^t \bigl(Y_s - Y_s^{(n)}\bigr)^2 \mathrm{d}\langle M\rangle_s \ \underset{n\to\infty}{\overset{P}{\longrightarrow}} \ 0. \tag{16.36}$$

Then also

$$\int_0^t Y_s^{(n)} \mathrm{d}M_s \ \underset{n\to\infty}{\overset{P}{\longrightarrow}} \ \int_0^t Y_s \mathrm{d}M_s. \tag{16.37}$$

We also have continuity in probability of $M \mapsto \int_0^t Y_s \mathrm{d}M_s$ in the following sense:

Lemma 16.11 *Let $\{M^{(n)}\}_{n\in\mathbb{N}} \in (\mathscr{M}_{\mathrm{loc}}^{\mathrm{cont}})^{\mathbb{N}}$ and $M \in \mathscr{M}_{\mathrm{loc}}^{\mathrm{cont}}$ suppose that $Y \in \mathcal{V}_M^{\mathrm{loc}} \cap \bigcap_{n\geqslant 1} \mathcal{V}_{M^{(n)}}^{\mathrm{loc}}$ is such that, for some $t \geqslant 0$,*

$$\int_0^u Y_s^2 \,\mathrm{d}\langle M^{(n)} - M\rangle_s \ \underset{n\to\infty}{\overset{P}{\longrightarrow}} \ 0 \tag{16.38}$$

Then also

$$\int_0^t Y_s \mathrm{d}M_s^{(n)} \ \underset{n\to\infty}{\overset{P}{\longrightarrow}} \ \int_0^t Y_s \mathrm{d}M_s. \tag{16.39}$$

The proof of these lemmas proceeds by suitable localization. We leave the details to a homework exercise.

Further reading: Sections 3.1-3.3 of Karatzas-Shreve