

14. CONTINUOUS LOCAL MARTINGALES

In this section we take the concepts discussed earlier to another level and generalize them by replacing the standard Brownian motion by a general continuous (local) martingale M . Much of the previous theory carries over with little additional effort once we construct the quadratic variation process $\langle M \rangle$ which for Brownian motion was effectively done in Proposition 6.2. This rather difficult step will be performed here.

14.1 Doob-Meyer decomposition for squares of local martingales.

Earlier we introduced a concept of a martingale $\{M_t: t \geq 0\}$ with respect to a filtration $\{\mathcal{F}_t\}_{t \geq 0}$. We now extend this to:

Definition 14.1 (Local martingale) *A local martingale is a process $\{M_t: t \geq 0\}$ for which there is a sequence $\{T_n: n \geq 1\}$ of non-negative random variables such that*

- (1) $\forall n \geq 1: T_n$ is a stopping time and $\{M_{T_n \wedge t}: t \geq 0\}$ is a martingale, and
- (2) $T_n \rightarrow \infty$ a.s. as $n \rightarrow \infty$.

All the concepts above are defined relative to the same filtration. A local martingale is continuous if $t \mapsto M_t$ is continuous.

A proper martingale is of course a local martingale (take, e.g., $T_n := n$). Assuming a continuous version of stochastic integrals, a more salient example of a local martingale is the process

$$M_t := \int_0^t Y_s dB_s \quad \text{for } Y \in \mathcal{V}^{\text{loc}} \quad (14.1)$$

Indeed, setting

$$T_n := \inf \left\{ t \geq 0: \int_0^t Y_s^2 ds \geq n \right\} \quad (14.2)$$

we have

$$M_{T_n \wedge t} = \int_0^t Y_1 1_{\{T_n > s\}} dB_s \quad \text{a.s.} \quad (14.3)$$

which implies $M_{T_n \wedge t} \in L^2$ as well as that $\{M_{T_n \wedge t}: t \geq 0\}$ is a martingale. The condition $T_n \rightarrow \infty$ a.s. then follows from $\int_0^t Y_s^2 ds < \infty$ a.s. for each $t \geq 0$.

A somewhat more irregular example is the process defined as

$$M_t := B_{\sigma(t)} \quad (14.4)$$

where B is a standard Brownian motion and $\sigma: [0, \infty) \rightarrow [0, \infty)$ is a continuous, non-decreasing function with $\sigma(0) = 0$. The filtration can be taken as

$$\mathcal{F}_t := \sigma(B_s: s \leq \sigma(t)) \quad (14.5)$$

The main point here is that the assumptions allow us to take σ quite irregular; e.g., a Cantor function which "moves" only on a set of vanishing Lebesgue measure of arguments. The identity $E((M_t - M_s)^2) = \sigma(t)^2 - \sigma(s)^2$ shows that so will then "move" the process M giving us (with the help of conditional Itô isometry) an example of a continuous local martingale that falls outside of the class (14.1)

With M a local martingale, the process $\{M_t^2 : t \geq 0\}$ is a local submartingale (with the term defined as above except for requiring that $\{M_{T_n \wedge t} : t \geq 0\}$ is a submartingale). For the example $M_t := B_t$ which for B denoting a standard Brownian motion is a proper martingale, we know that $\{B_t^2 - t : t \geq 0\}$ is again a martingale. A statement of this form actually holds for all continuous local martingales:

Theorem 14.2 *Let $\{M_t : t \geq 0\}$ be a continuous local martingale with respect to filtration $\{\mathcal{F}_t\}_{t \geq 0}$ such that \mathcal{F}_0 contains all P -null sets. Then there exists an adapted, continuous, non-decreasing process $\{\langle M \rangle_t : t \geq 0\}$ such that $\langle M \rangle_0 = 0$ and*

$$\{M_t^2 - \langle M \rangle_t : t \geq 0\} \text{ is a local martingale} \tag{14.6}$$

Any two processes with all these properties are indistinguishable.

We note that this result is a special instance of the *Doob-Meyer decomposition* of a continuous submartingale (of which $\{M_t^2 : t \geq 0\}$ is an example) as the sum of a continuous local martingale and a non-decreasing continuous process. The proof of that result is quite technical and so we decided to specialize it to local submartingales that take the form of the square of a local martingale. After all, this is all that we will ever need in the applications to stochastic integration.

14.2 Proof of uniqueness.

We will start with the proof of uniqueness which, as it turns out, also enters part of the existence argument. This relies on:

Lemma 14.3 *Let $t \geq 0$ and assume $\{M_s : s \geq 0\}$ is a continuous local martingale whose a.e. sample path is of bounded variation on $[0, t]$; i.e., $V_t^{(1)}(M) := \sup_{\Pi} V_t^{(1)}(M, \Pi) < \infty$ a.s. where the supremum is over partitions Π on $[0, t]$. Then*

$$P(\forall s \leq t : M_s = M_0) = 1 \tag{14.7}$$

meaning that M is constant a.s.

Proof. Since M is continuous, also $s \mapsto V_s^{(1)}(M)$ is continuous and non-decreasing on the set where it is finite. The random variable

$$T_n := \inf\{s \geq 0 : |M_s| \geq n \vee V_s^{(1)}(M) \geq n\} \tag{14.8}$$

is thus a stopping time such that $\widetilde{M}_s := M_{T_n \wedge s}$ is a martingale that is a bounded, $\sup_{s \geq 0} |\widetilde{M}_s| \leq n$, and has bounded variation, $\sup_{s \geq 0} V_s^{(1)}(\widetilde{M}) \leq n$. In particular, \widetilde{M} is an L^2 -martingale. For each $s \geq 0$ and any partition Π of $[0, s]$, the fact that the increments of \widetilde{M} are uncorrelated gives

$$\begin{aligned} E(|\widetilde{M}_s - \widetilde{M}_0|^2) &= E(V_s^{(2)}(\widetilde{M}, \Pi)) \leq E\left(\text{osc}_{\widetilde{M}}([0, t], \|\Pi\|) V_s^{(1)}(\widetilde{M}, \Pi)\right) \\ &\leq E\left(\text{osc}_{\widetilde{M}}([0, t], \|\Pi\|) V_s^{(1)}(\widetilde{M})\right) \leq n E\left(\text{osc}_{\widetilde{M}}([0, t], \|\Pi\|)\right) \end{aligned} \tag{14.9}$$

By continuity and boundedness of \widetilde{M} , the right-hand side tends to zero as $\|\Pi\| \rightarrow 0$ thanks to the Bounded Convergence Theorem. This gives $P(M_{s \wedge T_n} = M_0) = 1$ for each $s \geq 0$ and, taking countable unions,

$$\forall n \geq 1: P(\forall s \in \mathbb{Q} \cap [0, t \wedge T_n]: M_s = M_0) = 1. \quad (14.10)$$

The continuity of M allows us to drop the restriction to rational s . Since $P(T_n \leq t) \rightarrow 0$ a.s. as $n \rightarrow \infty$ by our assumptions, we get (14.7) as desired. \square

Proof of uniqueness in Theorem 14.2. Suppose A, A' are adapted non-decreasing continuous processes with $A_0 = A'_0 = 0$ such that both $\{M_t^2 - A_t: t \geq 0\}$ and $\{M_t^2 - A'_t: t \geq 0\}$ are local martingales. Then also $\{A_t - A'_t: t \geq 0\}$ is a local martingale whose every path is of bounded variation on $[0, t]$, for all $t \geq 0$. Hence $A_t - A'_t = A_0 - A'_0 = 0$ for all $t \geq 0$ a.s. meaning that $\{A_t: t \geq 0\}$ and $\{A'_t: t \geq 0\}$ are indistinguishable. \square

Note that continuity is essential for the uniqueness part to be true. Indeed, if $\{B_t: t \geq 0\}$ is a standard Brownian motion, then $\{B_t^2 - t: t \geq 0\}$ is a martingale. But so is the stochastic process $\{B_t^2 - N_t: t \geq 0\}$ if $\{N_t: t \geq 0\}$ is a Poisson process independent of the Brownian motion. However, while of bounded variation, $\{N_t: t \geq 0\}$ is not continuous and so cannot serve as the process $\langle B \rangle$. (Notably, the conclusion of Lemma 14.3 fails for $\{N_t: t \geq 0\}$ as well.)

The uniqueness clause already gives us existence for continuous local martingales arising from stochastic integrals:

Corollary 14.4 *Assume \mathcal{F}_0 contains all P -null sets. For the continuous local martingale $M_t := \int_0^t Y_s dB_s$ with $Y \in \mathcal{V}^{\text{loc}}$ and a continuous version of the integral assumed, we can take*

$$\forall t \geq 0: \langle M \rangle_t := \int_0^t Y_s^2 ds \quad (14.11)$$

on the full measure event $\{\forall t \geq 0: \int_0^t Y_s^2 ds < \infty\}$ and $M_t := 0$ otherwise.

Proof. In Corollary 12.4 we checked that, for $Y \in \mathcal{V}$, the process in (14.11) is exactly what reduces the square of the stochastic integral to a martingale. For $Y \in \mathcal{V}^{\text{loc}}$ we localize using the stopping time $T_n := \inf\{t \geq 0: \int_0^t Y_s^2 ds \geq n\}$ and note that, relying also on $\{Y_s 1_{\{T_n > s\}}: s \geq 0\} \in \mathcal{V}$ and Theorem 11.9, $M_{T_n \wedge t} = \int_0^t Y_s 1_{\{T_n > s\}} dB_s$ and $\langle M \rangle_{T_n \wedge t} = \int_0^t (Y_s 1_{\{T_n > s\}})^2 ds$ give that $\{M_{T_n \wedge t}^2 - \langle M \rangle_{T_n \wedge t}: t \geq 0\}$ is a martingale. Since $T_n \rightarrow \infty$ a.s. as $n \rightarrow \infty$, the claim holds for $Y \in \mathcal{V}^{\text{loc}}$ as well. \square

The uniqueness clause also gives:

Lemma 14.5 *The processes U and Y giving rise to the diffusion X via (13.10) are determined uniquely from X up to modification on a Lebesgue-null set of times. Explicitly, if U, Y and $\widetilde{U}, \widetilde{Y}$ are pairs of processes obeying (1-2) of Definition (13.1) and, assuming continuous versions of the integrals, for some $t \geq 0$ we have*

$$\forall u \leq t: \int_0^u U_s ds + \int_0^u Y_s dB_s = \int_0^u \widetilde{U}_s ds + \int_0^u \widetilde{Y}_s dB_s \quad \text{a.s.} \quad (14.12)$$

then also

$$\lambda(s \in [0, t]: U_s \neq \tilde{U}_s \vee Y_s \neq \tilde{Y}_s) = 0 \quad \text{a.s.} \quad (14.13)$$

where λ is the Lebesgue measure.

Proof. From (14.12) we get

$$\forall u \leq t: \int_0^u (Y_s - \tilde{Y}_s) dB_s = \int_0^u (\tilde{U}_s - U_s) ds \quad \text{a.s.} \quad (14.14)$$

But the left-hand side is a local martingale and, by (13.2), the right-hand side is of bounded variation on $[0, t]$ a.s. Using Lemma 14.3 it follows that the left-hand side vanishes a.s. and then so does the right-hand side. Lemma 12.8 and Corollary 14.4 gives $\int_0^t |Y_s - \tilde{Y}_s|^2 ds = 0$ a.s. and $\forall u \leq t: \int_0^u (\tilde{U}_s - U_s) ds = 0$ a.s. By the Lebesgue differentiation theorem, $\{s \in [0, t]: Y_s \neq \tilde{Y}_s \vee U_s \neq \tilde{U}_s\}$ has zero Lebesgue measure a.s. \square

14.3 Key lemmas for existence.

We now move to the proof of existence of $\langle M \rangle$ which, as for Brownian motion, relies on proving the existence of the limit of $V_2^{(2)}(M, \Pi)$ as $\|\Pi\| \rightarrow 0$. We first prove:

Lemma 14.6 *Let $t \geq 0$ and let M be a continuous martingale such that, for some $K \geq 0$, we have $\sup_{0 \leq s \leq t} |M_s| \leq K$ a.s. Then for any two partitions Π and $\tilde{\Pi}$ of $[0, t]$ with $\|\tilde{\Pi}\| \leq \|\Pi\|$,*

$$E \left(\left| V_t^{(2)}(M, \tilde{\Pi}) - V_t^{(2)}(M, \Pi) \right|^2 \right) \leq 112 K^2 \left[E \left(\text{osc}_M([0, t], \|\Pi\|)^4 \right) \right]^{1/2} \quad (14.15)$$

Proof. Fix $t \geq 0$ and let Π and $\tilde{\Pi}$ be partitions of $[0, t]$. Suppose first that $\tilde{\Pi}$ is a subpartition of Π . Specifically, assume Π is given by partition points

$$0 = t_{1,0} < t_{2,0} < \cdots < t_{n+1,0} = t \quad (14.16)$$

and $\tilde{\Pi}$ by partition points

$$0 = t_{1,0} < \cdots < t_{1,m_1} = t_{2,0} < \cdots < t_{n-1,m_{n-1}} = t_{n,0} < \cdots < t_{n,m_n} = t_{n+1,0} = t \quad (14.17)$$

where the first index marks the interval of partition Π where the corresponding subinterval of $\tilde{\Pi}$ marked by the second index lands into. Denoting

$$Z_{i,j} := M_{t_{i,j}} - M_{t_{i,j-1}} \quad (14.18)$$

hereby we get

$$V_t^{(2)}(M, \tilde{\Pi}) = \sum_{i=1}^n \sum_{j=1}^{m_i} Z_{i,j}^2 \quad (14.19)$$

while

$$V_t^{(2)}(M, \Pi) = \sum_{i=1}^n \left(\sum_{j=1}^{m_i} Z_{i,j} \right)^2 \quad (14.20)$$

Hence

$$V_t^{(2)}(M, \Pi) - V_t^{(2)}(M, \tilde{\Pi}) = 2 \sum_{i=1}^n \sum_{1 \leq j < j' \leq m_i} Z_{i,j} Z_{i,j'} \quad (14.21)$$

Next we want to square the expression and take expectation. This leads to the sum of expectations of the form $E(Z_{i,j} Z_{i,j'} Z_{i',k} Z_{i',k'})$, where $j < j'$ and $k < k'$. (These are meaningful because M and thus the Z 's are bounded.) A key observation is that, thanks to the Z 's being martingale increments, the expectation vanishes unless $i = i'$ and $k' = j'$. The expectation on the left of (14.15) thus equals

$$\begin{aligned} 4 \sum_{i=1}^n \sum_{1 < j' \leq m_i} \sum_{1 \leq j, k < j'} E(Z_{i,j} Z_{i,k} Z_{i,j'}^2) \\ = 4 \sum_{i=1}^n \sum_{j'=2}^{m_i} E((M_{t_{i,j'-1}} - M_{t_{i,0}})^2 (M_{t_{i,j'}} - M_{t_{i,j'-1}})^2) \end{aligned} \quad (14.22)$$

We can now bound the first difference on the right by the oscillation $\text{osc}_M([0, t], \|\Pi\|)$ and resum the rest. This bounds the expression by

$$4E\left(\text{osc}_M([0, t], \|\Pi\|)^2 V_t^{(2)}(M, \tilde{\Pi})\right). \quad (14.23)$$

We now invoke the Cauchy-Schwarz inequality which leads us to bound the second moment of $V_t^{(2)}(M, \tilde{\Pi})$. Here we relabel the points of $\tilde{\Pi}$ as $0 = s_0 < \dots < s_r = t$ and write

$$\begin{aligned} E(V_t^{(2)}(M, \tilde{\Pi})^2) &= E(V_t^{(4)}(M, \tilde{\Pi})^2) + 2 \sum_{i=1}^r E((M_{s_i} - M_{s_{i-1}})^2 (M_t - M_{s_i})^2) \\ &\leq 4K^2 E(V_t^{(2)}(M, \tilde{\Pi})^2) + 8K^2 E(V_t^{(2)}(M, \tilde{\Pi})^2) \\ &= 12K^2 E((M_t - M_0)^2) \leq 48K^4 \leq (7K^2)^2 \end{aligned} \quad (14.24)$$

Plugging this above we get the claim with 28 instead of 112 assuming, however, that $\tilde{\Pi}$ is a refinement of Π . To address the general case, consider a common refinement of the two partitions and relate each partition to that using the triangle inequality and the inequality $(a + b)^2 \leq 2a^2 + 2b^2$. \square

Using the estimate proved in the previous lemma, we now construct a candidate for $\langle M \rangle$ for uniformly bounded continuous martingales:

Lemma 14.7 *Assume that \mathcal{F}_0 contains all P -null sets and let $\{M_t : t \geq 0\}$ be a martingale such that, for some $K > 0$, we have $\sup_{t \geq 0} |M_t| \leq K$ a.s. Then there exists a non-decreasing continuous adapted process $\{\langle M \rangle_t : t \geq 0\}$ with $\langle M \rangle_0 = 0$ such that for any $t \geq 0$ and any sequence $\{\Pi_n\}_{n \geq 0}$ of partitions of $[0, t]$,*

$$\|\Pi_n\| \rightarrow 0 \quad \Rightarrow \quad V_t^2(M, \Pi_n) \xrightarrow[n \rightarrow \infty]{L^2} \langle M \rangle_t. \quad (14.25)$$

Moreover, $E(\langle M \rangle_t) \leq 4K^2$ holds for all $t \geq 0$.

Proof. We start with some general considerations. First, for the sake of this proof, a partition of \mathbb{R}_+ refers to any sequence of the form $\{0 = t_0 < t_1 < \dots < t_n < \infty\}$ with n finite. We will continue using $\|\Pi\| := \max_{i=1, \dots, n} |t_i - t_{i-1}|$. The symbol $V_t^{(2)}(M, \Pi)$ then refers to the usual formula with Π restricted to $[0, t]$ — which amounts to inserting t into the sequence and dropping all t_i 's outside $[0, t]$.

Now let $0 \leq u \leq s \leq t$ and observe that M being an L^2 -martingale implies

$$E((M_t - M_u)^2 | \mathcal{F}_s) = (M_s - M_u)^2 + E((M_t - M_s)^2 | \mathcal{F}_s). \quad (14.26)$$

It follows that if Π is a partition of \mathbb{R}_+ and $0 \leq s \leq t$, then

$$E(V_t^{(2)}(M, \Pi) | \mathcal{F}_s) = V_s^{(2)}(M, \Pi) + E((M_t - M_s)^2 | \mathcal{F}_s), \quad (14.27)$$

Denoting

$$A_s^\Pi := V_s^{(2)}(M, \Pi). \quad (14.28)$$

for any two partitions Π and $\tilde{\Pi}$ of \mathbb{R}_+ and any $0 \leq s \leq t$ we then have

$$A_s^\Pi - A_s^{\tilde{\Pi}} = E(A_t^\Pi - A_t^{\tilde{\Pi}} | \mathcal{F}_s) \quad (14.29)$$

proving that

$$\{A_s^\Pi - A_s^{\tilde{\Pi}} : s \geq 0\} \text{ is a martingale w.r.t. } \{\mathcal{F}_s\}_{s \geq 0} \quad (14.30)$$

The continuity of M implies that $s \mapsto A_s^\Pi - A_s^{\tilde{\Pi}}$ is actually continuous as well.

Let Π and $\tilde{\Pi}$ be two partitions of \mathbb{R}_+ . In light of (14.30) and continuity, Doob's L^2 -maximal inequality (10.3) gives

$$E\left(\sup_{s \leq t} |A_s^\Pi - A_s^{\tilde{\Pi}}|^2\right) \leq 4E(|A_t^\Pi - A_t^{\tilde{\Pi}}|^2). \quad (14.31)$$

The right-hand side is bounded using (14.15). Let $\{\Pi_n\}_{n \geq 0}$ be a sequence of partitions of \mathbb{R}_+ such that $n \rightarrow \|\Pi_n\|$ is non-increasing and

$$E\left(\text{osc}_M([0, n], \|\Pi_n\|)^4\right) \leq 64^{-n} \quad (14.32)$$

(This is possible because the $E(\text{osc}_M([0, n], \delta)^4) \rightarrow 0$ tends to zero as $\delta \rightarrow 0$ by the Bounded Convergence Theorem.) Using the Chebyshev inequality along with (14.15) we then get

$$P\left(\sup_{t \leq n} |A_t^{\Pi_{n+1}} - A_t^{\Pi_n}| > 2^{-n}\right) \leq 4 \cdot 112 K^2 2^{-n} \quad (14.33)$$

Denoting

$$\Omega^* := \Omega \setminus \left\{ \sup_{t \leq n} |A_t^{\Pi_{n+1}} - A_t^{\Pi_n}| > 2^{-n} \text{ i.o.} \right\} \quad (14.34)$$

Borel-Cantelli lemma gives $P(\Omega^*) = 1$ and

$$\forall t \geq 0: A_t := \lim_{n \rightarrow \infty} A_t^{\Pi_n} \text{ exists on } \Omega^* \quad (14.35)$$

with the limit locally uniform. On $\Omega \setminus \Omega^*$, we set $A_t := 0$ for $t \geq 0$. Since A^Π is continuous, the fact that the limit is uniform implies that A is continuous.

The inequality (14.15) shows that $V_t^{(2)}(M, \Pi) \rightarrow A_t$ in probability and in L^2 whenever $\|\Pi\| \rightarrow 0$. The bound $E(A_t) \leq 4K^2$ follows from

$$E(V_t^{(2)}(M, \Pi)) = E((M_t - M_0)^2) \leq 4K^2 \quad (14.36)$$

and Fatou's lemma. Since \mathcal{F}_0 contains all P -null sets, the construction implies that the process A is adapted. For a partition Π of $[0, t]$ containing s , we also have

$$A_t^\Pi - A_s^\Pi = V_t^{(2)}(M, \Pi) - V_s^{(2)}(M, \Pi') \geq 0 \quad (14.37)$$

and so, taking the L^2 -limit, we get $A_s \leq A_t$ a.s. for each $s \leq t$. Denoting $\Omega_0 := \bigcap_{s, t \in \mathbb{Q}, s < t} \{A_s \leq A_t\}$, for each $t \geq 0$ we now set

$$\langle M \rangle_t := \begin{cases} A_t, & \text{on } \Omega_0, \\ 0, & \text{else.} \end{cases} \quad (14.38)$$

Then $\langle M \rangle$ is continuous and non-decreasing on rationals and so non-decreasing everywhere. Since $P(\Omega_0) = 1$, the process is $\langle M \rangle$ also adapted. The limit statement (14.25) holds thanks to $\langle M \rangle_t = A_t$ a.s. \square

14.4 Proof of existence.

Equipped with above observations, we are now ready to give:

Proof of Theorem 14.2, existence. Let us first assume that $\{M_t : t \geq 0\}$ is a bounded martingale, meaning there exists $K \geq 0$ such that $\sup_{t \geq 0} |M_t| \leq K$ a.s. Let $\{\langle M \rangle_t : t \geq 0\}$ be the process constructed in Lemma 14.7. Then for each $0 \leq s \leq t$ and any partition Π containing s ,

$$E(M_t^2 | \mathcal{F}_s) - M_s^2 = E((M_t - M_s)^2 | \mathcal{F}_s) = E(V_t^{(2)}(M, \Pi) - V_s^{(2)}(M, \Pi') | \mathcal{F}_s) \quad (14.39)$$

where Π' is the restriction of Π to $[0, s]$. Taking $\{\Pi_n\}_{n \geq 0}$ with $\|\Pi_n\| \rightarrow 0$, the convergence from Lemma 14.7 gives

$$E(M_t^2 | \mathcal{F}_s) - M_s^2 = E(\langle M \rangle_t - \langle M \rangle_s | \mathcal{F}_s) \quad (14.40)$$

which using that $\langle M \rangle$ is adapted shows that $\{M_t^2 - \langle M \rangle_t : t \geq 0\}$ is a martingale.

Next assume only that $\{M_t : t \geq 0\}$ is a continuous local martingale. Define

$$\tau_K := \inf\{t \geq 0 : |M_t| \geq K\} \quad (14.41)$$

and let $\langle M^{(K)} \rangle$ be the unique non-decreasing, continuous and adapted process with $\langle M^{(K)} \rangle_0 = 0$ such that $\{M_{\tau_K \wedge t}^2 - \langle M^{(K)} \rangle_t : t \geq 0\}$ is a martingale. Then for $L \geq K$, the process $\{\langle M^{(K)} \rangle_{\tau_K \wedge t} - \langle M^{(L)} \rangle_{\tau_K \wedge t} : t \geq 0\}$ is a continuous martingale of bounded variation, implying that it is constant a.s. by Lemma 14.3. Denote

$$\Omega_0 := \{\tau_K \rightarrow \infty\} \cap \bigcap_{K \geq 1} \bigcap_{L \geq K} \{\forall t \in [0, \tau_K] : \langle M^{(K)} \rangle_t = \langle M^{(L)} \rangle_t\} \quad (14.42)$$

and observe that $\lim_{K \rightarrow \infty} \langle M^{(K)} \rangle_t$ exists on Ω_0 due to the fact that the sequence is constant once $\tau_K \geq t$. We may thus define

$$\langle M \rangle_t := \begin{cases} \lim_{K \rightarrow \infty} \langle M^{(K)} \rangle_t, & \text{on } \Omega_0, \\ 0, & \text{else,} \end{cases} \quad (14.43)$$

and observe that $\langle M \rangle$ is non-decreasing and continuous with $\langle M \rangle_0 = 0$.

Since $\tau_K \rightarrow \infty$ a.s. by continuity of M , the above observations imply that $P(\Omega_0) = 1$ and so $\langle M \rangle$ is also adapted. The fact that $\langle M \rangle_{\tau_K \wedge t} = \langle M^{(K)} \rangle_{\tau_K \wedge t}$ for all $t \geq 0$ on Ω_0 now shows that $\{M_{\tau_K \wedge t}^2 - \langle M \rangle_{\tau_K \wedge t} : t \geq 0\}$ a martingale for each $K \geq 1$. Using $\tau_K \rightarrow \infty$ a.s. we conclude that $\{M_t^2 - \langle M \rangle_t : t \geq 0\}$ is a local martingale, as desired. \square

Here are some examples of quadratic variation processes. For M being the standard Brownian motion B , we have $\langle B \rangle_t = t$. Since the Brownian motion is a Gaussian process that has all moments, we conclude that $\{B_t^2 - t : t \geq 0\}$ is martingale. For the quadratic variation of this martingale we note that, by Itô formula,

$$B_t^2 - t = \int_0^t 2B_s dB_s \quad (14.44)$$

Hence we get

$$\langle B^2 - \text{id} \rangle_t = 4 \int_0^t B_s^2 ds \quad (14.45)$$

by Corollary 14.4.

Another example is the process

$$M_t := \exp \left\{ \int_0^t Y_s dB_s - \frac{1}{2} \int_0^t Y_s^2 ds \right\} \quad (14.46)$$

for $Y \in \mathcal{V}^{\text{loc}}$ for which the Itô formula gives

$$dM_t = M_t(Y_t dB_t - \frac{1}{2}Y_t^2 dt) + \frac{1}{2}M_t Y_t^2 dt = M_t Y_t dB_t \quad (14.47)$$

In integral form, this reads

$$M_t = 1 + \int_0^t M_s Y_s dB_s \quad (14.48)$$

showing that M is a local martingale. (Note that M generalizes the Brownian exponential martingale $\{e^{\lambda B_t - \frac{1}{2}\lambda^2 t} : t \geq 0\}$.) Here we get

$$\langle M \rangle_t = \int_0^t M_s^2 Y_s^2 ds \quad (14.49)$$

by calling upon Corollary 14.4.

Yet another class of example is the martingale from (14.4) with respect to the filtration in (14.5). Here $\langle M \rangle_t = \sigma(t)^2$. The point of this example is to note that neither the martingale nor the quadratic variation process may arise from integrals (which would be the case here only if σ^2 is an AC function).

Further reading: Chapter 1 of Karatzas-Shreve