11. Extension via localization

In the previous lectures we progressively constructed the Itô integral for every process that is jointly-measurable, adapted and square integrable with respect to both time and the underlying probability measure on any compact interval of times. While technically convenient for the construction, the requirement of square integrability is often a nuisance and so we will now show how to get even beyond that.

11.1 Statement and main proposition.

Our standing assumption throughout this sections are as follows: We are given a probability space $(\Omega, \mathcal{F}, P)$ that supports a standard Brownian motion $\{B_t : t \geqslant 0\}$ and a Brownian filtration $\{\mathcal{F}_t\}_{t\geqslant 0}$ such that $\mathcal{F}_0$ contains all P-null sets. Let $\mathcal{V}^{\text{loc}}$ denote the set of processes $Y = \{Y_t : t \geqslant 0\}$ on $(\Omega, \mathcal{F}, P)$ that are jointly measurable, adapted to the filtration $\{\mathcal{F}_t\}_{t\geqslant 0}$ and obey

$$\forall t \geqslant 0\colon \quad \int_0^t Y_s^2 \, \mathrm{d}s < \infty \quad \text{a.s.} \tag{11.1}$$

where the integral is meaningful (as a Lebesgue integral) thanks to the assumption of joint measurability (which implies that $t \mapsto Y_t(\omega)$ is a Borel function for each $\omega \in \Omega$). The said extension now comes in:

Theorem 11.1 *Let $Y \in \mathcal{V}^{\text{loc}}$ and, for $M > 0$, set*

$$T^{(M)} := \inf\Big\{t \geqslant 0\colon \int_0^t Y_s^2 \, \mathrm{d}s \geqslant M\Big\} \tag{11.2}$$

Then for each $M > 0$,

(1) $\{Y_t 1_{\{T^{(M)}>t\}} : t \geqslant 0\} \in \mathcal{V}$, *and*
(2) *for all $t \geqslant 0$ and all $N \geqslant M$,*

$$\int_0^t Y_s 1_{\{T^{(N)}>s\}} \, \mathrm{d}B_s = \int_0^t Y_s 1_{\{T^{(M)}>s\}} \, \mathrm{d}B_s \quad \text{a.s. on } \{T^{(M)} > t\} \tag{11.3}$$

In particular, for each $t \geqslant 0$,

$$\int_0^t Y_s \, \mathrm{d}B_s := \lim_{M\to\infty} \int_0^t Y_s 1_{\{T^{(M)}>s\}} \, \mathrm{d}B_s \tag{11.4}$$

exists and is finite a.s. For $Y \in \mathcal{V}$ this coincides with the Itô integral defined previously.

The main difficulty in part (1) is to show that the process there is adapted and measurable. For this we recall the following concept:

Definition 11.2 *Given a filtration $\{\mathcal{F}_t\}_{t\geqslant 0}$, a $[0,\infty]$-valued random variable T is a* stopping time *(for that filtration) if*

$$\forall t \geqslant 0\colon \quad \{T \leqslant t\} \in \mathcal{F}_t \tag{11.5}$$

Note that, unlike discrete-time stopping times, we are not attempting to formulate this concept using the events $\{T = t\}$ as these are typically null and, anyway, cannot tell us anything about $\{T \leqslant t\}$ due to t taking uncountably many values. We now note:

Lemma 11.3 *Under the assumption that $\mathcal{F}_0$ contains all P-null sets, $T^{(M)}$ in* (11.2) *is a stopping time and, in particular,*

$$\big\{Y_t 1_{\{T^{(M)}>t\}} \colon t \geqslant 0\big\} \in \mathcal{V} \tag{11.6}$$

holds for each $M \geqslant 0$ and each $Y \in \mathcal{V}^{\rm loc}$.

Proof. A standard monotone class argument shows that $\int_0^u Y_s^2\,{\rm d}s$ is $\mathcal{F}_u$-measurable once Y is jointly measurable and adapted. By the continuity of $u \mapsto \int_0^u Y_s^2\,{\rm d}s$ on the set where it is finite, we have

$$\begin{aligned}\{T^{(M)} \leqslant t\} = \Big\{\int_0^t Y_s^2\,{\rm d}s \geqslant M\Big\} \\ \cup \bigg(\Big\{\int_0^t Y_s^2\,{\rm d}s < M\Big\} \cap \bigcap_{n\geqslant 1}\Big\{\int_0^{t+1/n} Y_s^2\,{\rm d}s = \infty\Big\}\bigg)\end{aligned} \tag{11.7}$$

The events in the giant intersections are P-null and so belong to $\mathcal{F}_0$ by assumption. The remaining events lie in $\mathcal{F}_t$ and so we get $\{T^{(M)} \leqslant t\} \in \mathcal{F}_t$.

The same decomposition implies that the process $\{1_{\{T^{(M)}>t\}} \colon t \geqslant 0\}$, and thus also that in (11.6), is adapted. To get that $\{1_{\{T^{(M)}>t\}} \colon t \geqslant 0\}$, and thus also the process in (11.6), is jointly measurable note that by a monotonce class argument,

$$\bigg\{(\omega, t) \in \Omega \times [0,\infty) \colon \int_0^t Y_s(\omega)^2{\rm d}s \in A\bigg\} \in \mathcal{F} \otimes \mathcal{B}([0,\infty)) \tag{11.8}$$

for each Borel subset of $[0,\infty]$. The above decomposition then shows that

$$\big\{(\omega, t) \in \Omega \times [0,\infty) \colon 1_{\{T^{(N)}(\omega)>t\}} = 0\big\} \in \mathcal{F} \otimes \mathcal{B}([0,\infty)) \tag{11.9}$$

as well. As $\int_0^t (Y_s 1_{\{T^{(M)}>s\}})^2{\rm d}s \leqslant M$ a.s. for all $t \geqslant 0$, the process in (11.6) is also uniformly square integrable and thus belongs to $\mathcal{V}$. □

The main argument of the proof of Theorem 11.2 now comes in:

Proposition 11.4 *Under the assumption that $\mathcal{F}_0$ contains all P-null sets, for each $Y \in \mathcal{V}$ and each stopping time T we have*

$$\big\{Y_t 1_{\{T>t\}} \colon t \geqslant 0\big\} \in \mathcal{V} \tag{11.10}$$

and, writing $\{I_t \colon t \geqslant 0\}$ for a continuous version of $t \mapsto \int_0^t Y_s {\rm d}B_s$, for each $t \geqslant 0$,

$$\int_0^{T\wedge t} Y_s\,{\rm d}B_s := I_{T\wedge t} = \int_0^t Y_s 1_{\{T>s\}}\,{\rm d}B_s \quad \text{a.s.} \tag{11.11}$$

Here the middle quantity is I_u evaluated at $u := T \wedge t$.

The proof of this proposition serves as a blueprint for many similar proofs involving stopping times. Indeed, we first check that the result holds for discrete-valued stopping times and simple processes and then take suitable limits to extend the conclusion to the stated general form.

11.2 Proof for discretized stopping times and simple processes.

We start with the discretization of T. For integer $N \geqslant 0$, define

$$T_N := 2^{-N}\lceil 2^N T\rceil. \tag{11.12}$$

which also means that $T_N = \infty$ if and only if $T = \infty$. We then have:

Lemma 11.5 *T_N is a stopping time with $T_N \downarrow T$ as $N \to \infty$. The processes $\{1_{\{T_N>t\}}\colon t \geqslant 0\}$ with $N \geqslant 0$, as well as $\{1_{\{T>t\}}\colon t \geqslant 0\}$, are adapted and jointly measurable.*

Proof. Since $T_N = k2^{-N}$ on $\{(k-1)2^{-N} < T \leqslant k2^{-N}\}$, we have

$$\{T_N \leqslant t\} = \bigcup_{k\leqslant 2^N t}\{(k-1)2^{-N} < T \leqslant k2^{-N}\} = \{T \leqslant 2^{-N}\lfloor 2^N t\rfloor\} \tag{11.13}$$

Since $2^{-N}\lfloor 2^N t\rfloor \leqslant t$ it follows that if T is a stopping time, then so is T_N. The dyadic nature of the discretization implies that $\{T_N\}_{N\geqslant 0}$ is non-increasing and, in light of

$$T_N - 2^{-N} \leqslant T \leqslant T_N \tag{11.14}$$

we get $T_N \downarrow T$ as $N \to \infty$ (including on $\{T = \infty\} = \{T_N = \infty\}$).

As $\{T_N\}_{N\geqslant 0}$ and T are stopping times, the processes $\{1_{\{T_N>t\}}\colon t \geqslant 0\}$ for any $N \geqslant 0$, as well as $\{1_{\{T>t\}}\colon t \geqslant 0\}$ are adapted. The joint measurability of $\{1_{\{T_N>t\}}\colon t \geqslant 0\}$ is checked by partitioning according to the values of T_N as

$$\big\{(\omega,t) \in \Omega\times[0,\infty)\colon 1_{\{T_N(\omega)>t\}=0}\big\} = \bigcup_{k\geqslant 0}\{T_N = k2^{-N}\}\times[k2^{-N},0) \tag{11.15}$$

where the containment of each event on the right in $\mathcal{F}\otimes\mathcal{B}([0,\infty))$ is implied by T_N being $\mathcal{F}$-measurable. The same argument yields joint measurability of $\{1_{\{T_N-2^{-N}>t\}}\colon t \geqslant 0\}$. But (11.14) shows that $1_{\{T_N-2^{-N}>t\}} \to 1_{\{T>t\}}$ for all $t \geqslant 0$ and so $\{1_{\{T>t\}}\colon t \geqslant 0\}$ is jointly measurable as well. □

Next, since $Y \in \mathcal{V}$, Theorem 10.5 tells us that there are $\{Y^{(n)}\}_{n\geqslant 0} \in \mathcal{V}_0^{\mathbb{N}}$ such that

$$[\![\, Y - Y^{(n)}]\!] \underset{n\to\infty}{\longrightarrow} 0. \tag{11.16}$$

We fix the sequence $\{Y^{(n)}\}_{n\geqslant 0}$ for the rest of the argument. The following lemma is where most of the work in the proof of Proposition 11.4 gets done.

Lemma 11.6 *For each $n \geqslant 1$ and $N \geqslant 0$,*

$$\big\{Y_t^{(n)} 1_{\{T_N>t\}}\colon t \geqslant 0\big\} \in \mathcal{V} \tag{11.17}$$

and, for each $t \geqslant 0$,

$$\int_0^{T_N\wedge t} Y_s^{(n)}\,\mathrm{d}B_s = \int_0^t Y_s^{(n)} 1_{\{T_N>s\}}\,\mathrm{d}B_s \quad \text{a.s.} \tag{11.18}$$

Here, on the left, we use the defining expression of the integral for processes in $\mathcal{V}_0$.

Proof. The containment $Y^{(n)} \in \mathcal{V}_0$ guarantees that each $Y^{(n)}$ is adapted, jointly measurable and bounded. Lemma 11.5 implies the same about $\{1_{\{T_N>t\}} \colon t \geqslant 0\}$ and thus also about $\{Y_t^{(n)} 1_{\{T_N>t\}} \colon t \geqslant 0\}$. This is enough to ensure containment in $\mathcal{V}$.

For the second part of the claim, note that for $t \geqslant 0$ the Tonelli Theorem applied to infinite sums along with the fact that T_N takes values in $2^{-N}\mathbb{N}$ yields

$$1_{\{T_N>t\}} = \sum_{k\geqslant 0} 1_{\{T_N>k2^{-N}\}} 1_{[k2^{-N},(k+1)2^{-N})}(t). \tag{11.19}$$

The expression on the right has almost the form of a simple process except for two points: The intervals are open/closed at opposite ends than what they should be for a simple process and, moreover, the sum is infinite rather than finite. The latter is resolved for $\{Y_t^{(n)} 1_{\{T_N>t\}} \colon t \geqslant 0\}$ by the fact that simple processes have *ex definitio* bounded support in time. For the former, we will have to work a bit.

First observe that we can always write Y in the form

$$Y_t^{(n)} = Z_0 1_{\{0\}}(t) + \sum_{i=1}^{m} Z_i 1_{(t_{i-1},t_i]}(t), \tag{11.20}$$

where $t_0 = 0 < t_1 < \dots < t_m$ and $Z_i \in L^\infty$ is $\mathcal{F}_{t_{i-1}}$-measurable for each i so that $t_m 2^N \in \mathbb{N}$ and $\{t_i \colon i = 0,\dots,t_m\}$ contains all the points in $\{k2^{-N} \colon k \geqslant 0 \wedge k2^{-N} \leqslant t_m\}$. Then

$$R_t := \sum_{i=1}^{m} Z_i 1_{\{T_N>t_{i-1}\}} 1_{(t_{i-1},t_i]}(t) \tag{11.21}$$

is a simple process. Moreover, when $t \in (t_{i-1}, t_i)$ we find the unique $k \geqslant 0$ so that $(t_{i-1}, t_i) \subseteq [k2^{-N}, (k+1)2^{-N})$ and note that $T_N > t$ is then equivalent to $T_N \geqslant 2^{-N}(k+1)$ which is equivalent to $T_N > t_{i-1}$. Hence we get

$$\bigl|Y_t^{(n)} 1_{\{T_N>t\}} - R_t\bigr| \leqslant \sum_{i=0}^{m} \|Z_i\| 1_{\{t_{i-1}\}}(t) \tag{11.22}$$

where $t_{-1} := 0$. Since Lebesgue integral of the right-hand side vanishes, the processes $\{Y_t^{(n)} 1_{\{T_N>t\}} \colon t \geqslant 0\}$ and $\{R_t \colon t \geqslant 0\}$ are equivalent in $\mathcal{V}$.

Using the defining expression for integrals of simple functions, we then get

$$\int_0^{T_N\wedge t} Y_s^{(n)}\, \mathrm{d}B_s = \sum_{i=1}^{m} Z_i (B_{T_N\wedge t\wedge t_i} - B_{T_N\wedge t\wedge t_{i-1}}) \tag{11.23}$$

The above-mentioned equivalence in turn gives

$$\int_0^t Y_s^{(n)} 1_{\{T_N>s\}}\, \mathrm{d}B_s \overset{\text{a.s.}}{=} \int_0^t R_s\, \mathrm{d}B_s = \sum_{i=1}^{m} Z_i 1_{\{T>t_{i-1}\}} (B_{t\wedge t_i} - B_{t\wedge t_{i-1}}) \tag{11.24}$$

Since T_N takes values in $\{t_i \colon i = 0,\dots,m\}$, we have

$$\begin{aligned} B_{T_N\wedge t\wedge t_i} - B_{T_N\wedge t\wedge t_{i-1}} &= \bigl(B_{T_N\wedge t\wedge t_i} - B_{T_N\wedge t\wedge t_{i-1}}\bigr) 1_{\{T_N\geqslant t_i\}} \\ &= \bigl(B_{t\wedge t_i} - B_{t\wedge t_{i-1}}\bigr) 1_{\{T_N\geqslant t_i\}} = \bigl(B_{t\wedge t_i} - B_{t\wedge t_{i-1}}\bigr) 1_{\{T_N>t_{i-1}\}} \end{aligned} \tag{11.25}$$

where the first equality follows from the fact that the difference on the left-hand side vanishes on $\{T_N < t_i\} = \{T_N \leqslant t_{i-1}\}$ and the rest is a rewrite based on the complementary equality $\{T_N \geqslant t_i\} = \{T_N > t_{i-1}\}$. This equates the right-hand sides of (11.24) with (11.23) proving (11.18) as desired. □

11.3 Taking limits.

Having proved the claim for simple processes and discretized stopping times, we now start taking limits, starting with the integral of the truncated process:

Lemma 11.7 *For each $t \geqslant 0$,*

$$\lim_{N\to\infty} \limsup_{n\to\infty} E\left(\left|\int_0^t Y_s^{(n)} 1_{\{T_N>s\}}\,\mathrm{d}B_s - \int_0^t Y_s 1_{\{T>s\}}\,\mathrm{d}B_s\right|^2\right) = 0. \tag{11.26}$$

Proof. Note that (11.14) implies

$$0 \leqslant 1_{\{T_N>s\}} - 1_{\{T>s\}} = 1_{\{T\leqslant s<T_N\}} \tag{11.27}$$

The triangle inequality combined with Itô isometry gives

$$\begin{aligned} &\|Y^{(n)}1_{\{T_N>\cdot\}} - Y1_{\{T>\cdot\}}\|_{L^2([0,t]\times\Omega)} \\ &\qquad \leqslant \|Y^{(n)} - Y\|_{L^2([0,t]\times\Omega)} + \left[\int_0^t Y_s^2 1_{\{T\leqslant s<T_N\}}\mathrm{d}s\right]^{1/2} \end{aligned} \tag{11.28}$$

The first term vanishes as $n \to \infty$ by (11.16). Using $1_{\{T\leqslant s<T_N\}} \to 1_{\{T=s\}}$ as $N \to \infty$, the Dominated Convergence Theorem shows that the second term vanishes as $N \to \infty$. In light of the isometry-based construction of the Itô integral, this implies (11.26). □

Next we address the limit of the integral truncated by the stopping time:

Lemma 11.8 *For each $t \geqslant 0$ and each $N \geqslant 0$,*

$$\int_0^{T_N\wedge t} Y_s^{(n)}\,\mathrm{d}B_s \xrightarrow[n\to\infty]{P} \int_0^{T_N\wedge t} Y_s\,\mathrm{d}B_s \tag{11.29}$$

where the integral on the right-hand side is to be interpreted interpreted as $I_{T_N\wedge t}$, for $\{I_t\colon t \geqslant 0\}$ denoting a continuous version of $\{\int_0^t Y_s \mathrm{d}B_s\colon t \geqslant 0\}$.

Proof. Pick $\epsilon > 0$ and note that, by Doob's L^2-inequality (see Lemma 10.3) relying on the fact that the stochastic integral is an L^2-martingale,

$$\begin{aligned} &P\left(\left|\int_0^{T_N\wedge t} Y_s^{(n)}\,\mathrm{d}B_s - \int_0^{T_N\wedge t} Y_s\,\mathrm{d}B_s\right| > \epsilon\right) \\ &\qquad \leqslant P\left(\sup_{0\leqslant u\leqslant t}\left|\int_0^u Y_s^{(n)}\,\mathrm{d}B_s - \int_0^u Y_s\,\mathrm{d}B_s\right| > \epsilon\right) \leqslant \frac{1}{\epsilon^2}\|Y^{(n)} - Y\|^2_{L^2([0,t]\times\Omega)} \end{aligned} \tag{11.30}$$

The right-hand side tends to zero by (11.16). □

We are now ready to give:

Proof of Proposition 11.4. The claim in (11.10) follows by the same arguments as those used in the proof of Lemma 11.6. To get (11.11), we now invoke (11.18) along with Lemmas 11.7–11.8 and the fact that

$$\int_0^{T_N\wedge t} Y_s\,\mathrm{d}B_s \underset{N\to\infty}{\longrightarrow} \int_0^{T\wedge t} Y_s\,\mathrm{d}B_s \quad \text{a.s.} \tag{11.31}$$

implied by $T_N \downarrow T$ and the fact that the stochastic integral admits a continuous version almost surely (see Theorem 10.2). □

11.4 Proof of the main localization claim.

We are now ready to move to the proof of our main claim:

Proof of Theorem 11.1. Let $Y \in \mathcal{V}^{\text{loc}}$. Part (1) was proved in Lemma 11.3. For (2) we note that, since $M \mapsto T^{(M)}$ is non-decreasing,

$$N \geqslant M \quad \Rightarrow \quad 1_{\{T^{(M)}>s\}} = 1_{\{T^{(M)}>s\}} 1_{\{T^{(N)}>s\}} \tag{11.32}$$

Proposition 11.4 then gives

$$\begin{aligned}\int_0^t Y_s 1_{\{T^{(M)}>s\}}\,\mathrm{d}B_s &\overset{N\geqq M}{=} \int_0^t Y_s 1_{\{T^{(N)}>s\}} 1_{\{T^{(M)}>s\}}\,\mathrm{d}B_s \overset{\text{a.s.}}{=} \int_0^{T^{(M)}\wedge t} Y_s 1_{\{T^{(N)}>s\}}\,\mathrm{d}B_s \\ &= \int_0^t Y_s 1_{\{T^{(N)}>s\}}\,\mathrm{d}B_s \quad \text{on } \{T^{(M)} > t\}\end{aligned} \tag{11.33}$$

This proves (11.3) as well as (11.4), for (a.s.) constant sequences have (a.s.) limits and that limit exists on $\bigcup_{M\geqslant 1}\{T^{(M)} > t\}$ which is a full-measure event by (11.2). □

As a consequence of Theorems 10.2, 10.5 and 11.1 and also Proposition 11.4 we get:

Theorem 11.9 *Suppose the Brownian filtration $\{\mathcal{F}_t\}_{t\geqslant 0}$ is such that $\mathcal{F}_0$ contains all P-null sets. Then for all $Y \in \mathcal{V}^{\text{loc}}$ the integral defined in* (11.4) *admits a continuous version that is adapted to $\{\mathcal{F}_t\}_{t\geqslant 0}$ and obeys*

$$\int_0^{T\wedge t} Y_s \mathrm{d}B_s = \int_0^t Y_s 1_{\{T>s\}} \mathrm{d}B_s \quad \text{a.s.} \tag{11.34}$$

for each $t \geqslant 0$ and each stopping time T with respect to $\{\mathcal{F}_t\}_{t\geqslant 0}$. In particular, if $\{T_N\}_{N\geqslant 0}$ is a sequence of stopping times such that $\{Y_t 1_{\{T_N>t\}} \colon t \geqslant 0\} \in \mathcal{V}$ for each $N \geqslant 0$ and $T_N \to \infty$ a.s., then for each $t \geqslant 0$,

$$\int_0^t Y_s 1_{\{T_N>s\}} \mathrm{d}B_s \underset{N\to\infty}{\longrightarrow} \int_0^t Y_s \mathrm{d}B_s \quad \text{a.s.} \tag{11.35}$$

Proof. That a continuous version of the extended Itô integral exists follows from the fact that on $\{T^{(M)} > t\}$, the map $u \mapsto \int_0^u Y_s \mathrm{d}B_s$ for $u \in [0,t]$ coincides with $u \mapsto \int_0^u Y_s 1_{\{T^{(M)}>s\}} \mathrm{d}B_s$ which admits a continuous version by Theorem 10.2. Proposition 11.4 then gives

$$\int_0^{T\wedge t} Y_s \mathrm{d}B_s = \int_0^t Y_s 1_{\{T^{(M)}>s\}} 1_{\{T>s\}} \mathrm{d}B_s \quad \text{a.s. on } \{T^{(M)} > t\} \tag{11.36}$$

Letting $\widetilde{T}^{(M)} := \inf\{u \geqslant 0\colon \int_0^u Y_s^2 1_{\{T>s\}} \mathrm{d}s \geqslant M\}$, we now check $\widetilde{T}^{(M)} \wedge T = T^{(M)} \wedge T$ and so

$$\int_0^t Y_s 1_{\{T^{(M)}>s\}} 1_{\{T>s\}} \mathrm{d}B_s = \int_0^t Y_s 1_{\{T>s\}} 1_{\{\widetilde{T}^{(M)}>s\}} \mathrm{d}B_s \tag{11.37}$$

The right-hand side equals the integral on the right of (11.34) on $\{T^{(M)} > t\}$, which is a subset of $\{\widetilde{T}^{(M)} > t\}$. Hence we get (11.34) on $\bigcup_{M\geqslant 0}\{T^{(M)} > t\}$ which is a full-measure set by the fact that $Y \in \mathcal{V}^{\mathrm{loc}}$. By (11.34), the object on the left-hand side of (11.35) equals the right-hand side once $T_N > t$. As $T_N \to \infty$ a.s., the equality holds a.s. □

Remark 11.10 The upshot of (11.35) is that the extension of the Itô integral to $Y \in \mathcal{V}^{\mathrm{loc}}$ in (11.4) can be done along *any* sequence $\{T_N\}_{N\geqslant 0}$ of stopping times such that $T_N \to \infty$ a.s. as $N \to \infty$ and $\{Y_t 1_{\{T_N>t\}}\colon t \geqslant 0\} \in \mathcal{V}$ for every $N \geqslant 0$.

The Itô integral has now been extended to all $Y \in \mathcal{V}^{\mathrm{loc}}$. A further extension can be attempted by assuming (11.2) (or having Y defined) only up to a stopping time. These extensions are handled by working with the *stopped process* $\{Y_{t\wedge T}\colon t \geqslant 0\}$ (which still needs to be assumed in $\mathcal{V}^{\mathrm{loc}}$). We will also see later that no extension beyond processes for which $t \mapsto Y_t$ is locally square integrable exists.

Further reading: Section 3.2D of Karatzas-Shreve