10. Continuous version and Itô integrability

The purpose of this lecture is to resolve two natural questions. The first one concerns a characterization of processes that we can actually integrate in the above sense. Thanks to Lemma 9.9 we know that all left-continuous processes $Y \in \mathcal{V}$ are included, but the point is to determine how large is the class of integrable processes in $\mathcal{V}$. The second question concerns the t-dependence of the integral $\int_0^t Y_s \mathrm{d}B_s$. Note that we constructed the integral for each t separately, and only modulo a null set, but if are to treat $\{\int_0^t Y_s \mathrm{d}B_s \colon t \in [0,\infty)\}$ as a process then we need to establish some minimal regularity of the integrals as a function of their upper limit.

10.1 Continuous version.

We start by addressing the second question. For this we recall the following definition:

Definition 10.1 *Let $\{\mathcal{F}_t\}_{t\geqslant 0}$ be a filtration. A process $\{M_t \colon t \geqslant 0\}$ is a martingale if it is adapted (i.e., $\forall t \geqslant 0\colon\ M_t$ is $\mathcal{F}_t$-measurable), integrable (meaning $\forall t \geqslant 0\colon\ M_t \in L^1$) and*

$$\forall t,s \geqslant 0\colon \quad E(M_{t+s}|\mathcal{F}_t) = M_t \text{ a.s.} \tag{10.1}$$

We say that the martingale is continuous if $t \mapsto M_t$ is continuous. A martingale is an L^2-martingale if $\forall t \geqslant 0\colon\ M_t \in L^2$.

We then have:

Theorem 10.2 *Suppose the Brownian filtration is such that $\mathcal{F}_0$ contains all P-null sets. For each $Y \in \overline{\mathcal{V}_0}^{[\![\cdot]\!]}$ there exists a continuous L^2-martingale $\{I_t \colon t \geqslant 0\}$ such that*

$$\forall t \geqslant 0\colon \quad P\Bigl(I_t = \int_0^t Y_s \, \mathrm{d}B_s\Bigr) = 1 \tag{10.2}$$

In particular, the process $\{\int_0^t Y_s \mathrm{d}B_s \colon t \geqslant 0\}$ of stochastic integrals admits a continuous version.

For the proof we need the following result from martingale theory:

Lemma 10.3 (Doob's L^2-inequality) *Let $\{M_t \colon t \geqslant 0\}$ be a continuous L^2-martingale. Then*

$$\forall t \geqslant 0 \, \forall \lambda > 0\colon \quad P\Bigl(\sup_{s\in[0,t]} |M_s| > \lambda\Bigr) \leqslant \frac{1}{\lambda^2} E(M_t^2) \tag{10.3}$$

Proof. Fix $t \geqslant 0$ and $\lambda > 0$. We will use the fact that the statement is known for discrete-time martingales. Indeed, given any set of points $\{t_i \colon i = 0,\dots,n\} \subseteq [0,t]$, which we may assume to obey $0 = t_0 < t_1 < \dots < t_n \leqslant t$, the process $\{M_{t_i}\}_{i=1}^n$ is an L^2-martingale. The discrete-time Doob L^2-inequality then reads

$$P\Bigl(\max_{i=0,\dots,n} |M_{t_i}| > \lambda\Bigr) \leqslant \frac{1}{\lambda^2} E(M_{t_n}^2) \leqslant \frac{1}{\lambda^2} E(M_t^2), \tag{10.4}$$

where we also used that $t \mapsto E(M_t^2)$ is non-decreasing. Refining the family of points to increase to all rationals in $[0,t]$ yields

$$P\Bigl(\sup_{s\in[0,t]\cap\mathbb{Q}} |M_s| > \lambda\Bigr) \leqslant \frac{1}{\lambda^2} E(M_t^2) \tag{10.5}$$

where the Monotone Convergence Theorem was used on the left-hand side to pass the limit inside the probability. As M has continuous paths, the supremum in (10.5) equals that in (10.3) and so we are done. □

Proof of Theorem 10.2. Since $Y \in \overline{\mathcal{V}_0}^{[\![\cdot]\!]}$, for each $n \geqslant 1$ there is $Y^{(n)} \in \mathcal{V}_0$ such that

$$[\![Y^{(n)} - Y]\!] \leqslant 2^{-n}. \tag{10.6}$$

Since $[\![\widetilde{Y}]\!] \geqslant 2^{-k}\min\{1, \|\widetilde{Y}\|_{L^2([0,k]\times\Omega)}\}$, for $n \geqslant \lceil t\rceil$ this implies

$$\|Y^{(n)} - Y\|_{L^2([0,t]\times\Omega)} \leqslant 2^{-n+\lceil t\rceil} \tag{10.7}$$

Define

$$I_t^{(n)} := \int_0^t Y_s^{(n)}\,\mathrm{d}B_s. \tag{10.8}$$

Then $t \mapsto I_t^{(n)}$ is continuous with $I_t^{(n)} \in L^2$ and $\mathcal{F}_t$-measurable. We claim:

Lemma 10.4 *$I^{(n)}$ is a continuous L^2-martingale.*

Proof. In light of the above, it remains to check the martingale property (10.1). Assuming $Y^{(n)}$ is given by the right-hand side of (9.7), this boils down to showing that, for any $t_i > t_{i-1} \geqslant 0$, any $t, s \geqslant 0$, any $Z_i \in L^2$ that is $\mathcal{F}_{t_{i-1}}$-measurable,

$$E\bigl(Z_i(B_{t_i\wedge(t+s)} - B_{t_{i-1}\wedge(t+s)})\,\big|\,\mathcal{F}_t\bigr) = Z_i(B_{t_i\wedge t} - B_{t_{i-1}\wedge t}) \tag{10.9}$$

For $t \geqslant t_i$ this is immediate from the fact that all of the random variables on the left are $\mathcal{F}_{t_i}$-measurable, while for $t \leqslant t_{i-1}$ we first condition on $\mathcal{F}_{t_{i-1}}$ and then realize that this conditional expectation vanishes, as does the right-hand side. For $t \in (t_{i-1}, t_i)$ we can move Z_i out of the conditional expectation. Then we write the Brownian increment on the left as

$$(B_{t_i\wedge(t+s)} - B_t) + (B_{t\wedge t_i} - B_{t\wedge t_{i-1}}) \tag{10.10}$$

The conditional expectation (given $\mathcal{F}_t$) of the first term in the parentheses vanishes thanks to Definition 9.1(3), while the second term is $\mathcal{F}_t$-measurable and thus unaffected by taking this conditional expectation. Hence we get (10.9) in all cases. □

Returning to the proof of Theorem 10.2, we now observe that Doob's L^2-martingale inequality along with Itô isometry and (10.7) show, for any $m \geqslant n$, that

$$\begin{aligned} P\Bigl(\sup_{s\in[0,t]} |I_s^{(n)} - I_s^{(m)}| > 2^{-n/2}\Bigr) &\leqslant 2^n E\bigl[(I_t^{(n)} - I_t^{(m)})^2\bigr] \\ &= 2^n \|Y^{(n)} - Y^{(m)}\|_{L^2([0,t]\times\Omega)}^2 \leqslant 4\cdot 2^{-m\wedge n+2\lceil t\rceil} \end{aligned} \tag{10.11}$$

where the factor 4 arises from writing the norm on the left using the norms of $Y^{(n)} - Y$ and $Y^{(m)} - Y$ and bounding each separately. For $m := n+1$ this is summable on n and so, by the Borel-Cantelli lemma, the event on the left occurs only finitely often a.s. for each $t \geqslant 0$. Setting

$$\Omega_0 := \bigcap_{N\geqslant 1} \Big\{ \sup_{s\in[0,N]} |I_s^{(n)} - I_s^{(n+1)}| > 2^{-n/2} \text{ i.o.}(n) \Big\}^{\mathrm{c}} \tag{10.12}$$

we thus get $P(\Omega_0) = 1$. Now define

$$I_t := \begin{cases} \lim\limits_{n\to\infty} I_t^{(n)} & \text{on } \Omega_0 \\ 0 & \text{else,} \end{cases} \tag{10.13}$$

and note that the limit exist on the event in the first alternative, which by the previous reasoning is a full measure event.

The limit in (10.13) is actually locally uniform and, since each $I^{(n)}$ is continuous, also the limit process I is continuous. The construction of the Itô integral gives

$$\forall t \geqslant 0\colon \qquad I_t^{(n)} \underset{n\to\infty}{\overset{L^2}{\longrightarrow}} \int_0^t Y_s \mathrm{d}B_s \tag{10.14}$$

The uniqueness of L^2-limit along with the fact that it implies a.e. subsequential convergence then gives (10.2) as well as $I_t^{(n)} \to I_t$ in L^2. Since $\mathcal{F}_0$ contains all P-null sets, we have $\Omega_0, \Omega_0^{\mathrm{c}} \in \mathcal{F}_0$ and the process I is adapted. Lemma 10.4 shows that $I^{(n)}$ obeys $E(I_{t+s}^{(n)}|\mathcal{F}_t) = I_t^{(n)}$ a.s. The L^2-convergence $I_t^{(n)} \to I_t$ allows for taking $n \to \infty$ limit in the conditional expectation which shows that I is an L^2-martingale. □

10.2 Characterizing integrable processes.

As to the other question discussed above, here we simply claim:

Theorem 10.5 *Using the setting and notation defined above,*

$$\overline{\mathcal{V}_0}^{[\![\cdot]\!]} = \mathcal{V} \tag{10.15}$$

In particular, $\int_0^t Y \mathrm{d}B_s$ *is defined as in Corollary 9.8 for all jointly-measurable adapted locally-square integrable processes and all* $t \geqslant 0$*.*

Proof. Let $Y \in \mathcal{V}$. We need to show that Y can be approximated by a sequence of processes from $\mathcal{V}_0$. In Lemma 9.9 we used the values of Y at dyadic rationals to give a good approximation for Y left-continuous. Here we will use a similar idea but for dyadic rationals shifted by a random value so that a.e. shift actually produces a good approximation. The proof is thus an example of a *probabilistic method* which is a way to demonstrate existence of an object by showing that a sample from a natural probability measure yields a desired object with probability one.

Moving to the actual argument, for technical reasons we first extend $t \mapsto Y_t$ to all reals by setting $Y_t = 0$ for all $t < 0$. Define, for each integer $n \geqslant 1$ and reals $t, h \geqslant 0$,

$$r_n(t,h) := 2^{-n}\lfloor 2^n(t-h)\rfloor + h \tag{10.16}$$

Then $t \mapsto r_n(t,h)$ takes only values in $2^{-n}\mathbb{Z} + h$ and obeys

$$t - 2^{-n} \leqslant r_n(t,h) \leqslant t \tag{10.17}$$

Note also that $h \mapsto r_n(t,h)$ is a piecewise linear (but discontinuous) periodic function with period 2^{-n}. In particular, for each integer $n \geqslant 1$,

$$\int_0^1 f\big(r_n(t,h)\big)\mathrm{d}h = 2^n \int_0^{2^{-n}} f(t-h)\mathrm{d}h \tag{10.18}$$

holds for each Borel $f\colon \mathbb{R} \to [0,\infty)$ and each $t \geqslant 0$. This identity drives the proof of:

Lemma 10.6 *For all $T > 0$,*

$$E\int_0^T \mathrm{d}t \int_0^1 \mathrm{d}h\, |Y_t - Y_{r_n(t,h)}|^2 \underset{n\to\infty}{\longrightarrow} 0 \tag{10.19}$$

Proof. Fix $T > 0$. We first claim that, whenever $\int_0^T Y_t(\omega)^2 \mathrm{d}t < \infty$, we have

$$\int_0^T \big|Y_t(\omega) - Y_{t-h}(\omega)\big|^2\, \mathrm{d}t \underset{h\downarrow 0}{\longrightarrow} 0 \tag{10.20}$$

This holds because, for each $\omega \in \Omega$, the map $t \mapsto Y_t(\omega)$ — being measurable and square integrable on $[0,T]$ thanks to $Y \in \mathcal{V}$ — can be approximated in $L^2([0,T])$ by a sequence of bounded continuous functions for which the claim holds by the Bounded Convergence Theorem. (This argument is generally used to prove that the shift-map is continuous in L^p for every $p \in [1,\infty)$.)

For $h > 0$, the quantity in (10.20) is bounded by $4\int_0^T Y_t^2 \mathrm{d}t$, which has finite expectation by $Y \in \mathcal{V}$, the Dominated Convergence Theorem yields

$$E\int_0^T |Y_t - Y_{t-h}|^2\, \mathrm{d}t \underset{h\downarrow 0}{\longrightarrow} 0 \tag{10.21}$$

Noting that (10.18) along with Fubini-Tonelli gives

$$E\int_0^T \mathrm{d}t \int_0^1 \mathrm{d}h\, |Y_t - Y_{r_n(t,h)}|^2 = 2^n \int_0^{2^{-n}} \Big(E\int_0^T |Y_t - Y_{t-h}|^2\, \mathrm{d}t\Big)\mathrm{d}h, \tag{10.22}$$

the claim follows by plugging (10.21) on the right-hand side. □

Moving back to the main line of the proof of Theorem 10.5, using Lemma 10.6 along with the Cantor diagonal argument we now find an increasing sequence of integers $\{n_k\}_{k\in\mathbb{N}}$ such that

$$\forall k \in \mathbb{N}\colon\ E\int_0^k \mathrm{d}t \int_0^1 \mathrm{d}h\, |Y_t - Y_{r_{n_k}(t,h)}|^2 \leqslant 4^{-k} \tag{10.23}$$

The Markov inequality along with Fubini-Tonelli implies

$$\lambda\bigg(\Big\{h \in [0,1]\colon E\int_0^k \mathrm{d}t\, |Y_t - Y_{r_{n_k}(t,h)}|^2 > 2^{-k}\Big\}\bigg) \leqslant 2^{-k}, \tag{10.24}$$

where λ is the Lebesgue measure on $[0,1]$. The Borel-Cantelli lemma now implies that there exists $h \in [0,1]$ (in fact, there exists a full-measure set of such h's) such that

$$E\int_0^m \mathrm{d}t\,|Y_t - Y_{r_{n_k}(t,h)}|^2 \underset{k\to\infty}{\longrightarrow} 0 \tag{10.25}$$

holds for all $m \geqslant 1$.

To conclude, fix h where (10.25) holds. The piece-wise constant nature of $t \mapsto r_n(t,h)$ then implies that, for each $k \geqslant 1$, the process $Y^{(k)}$ defined by

$$Y_t^{(k)} := Y_{r_{n_k}(t,h)} 1_{[0,k]}(t) \tag{10.26}$$

obeys $Y^{(k)} \in \mathcal{V}_0$, thanks to the piece-wise constant nature of $t \mapsto r_n(t,h)$. Since (10.25) gives $[\![Y - Y^{(k)}]\!] \to 0$ as $k \to \infty$, we conclude that $Y \in \overline{\mathcal{V}_0}^{[\![\cdot]\!]}$ thus proving the claim. $\square$

Further reading: Section 3.1 in Øksendal, Section 3.2B in Karatzas-Shreve