HW#7: due Fri 12/5/2025

The homework ordeal is completed for the quarter by the following three problems on stochastic differential equations.

Problem 1: Find an explicit strong solution to the SDE

$$\mathrm{d}X_t = \left(\sqrt{1+X_t^2} + \tfrac{1}{2}X_t\right)\mathrm{d}t + \sqrt{1+X_t^2}\,\mathrm{d}B_t$$

Hint: Try to interpret the right-hand side via the Itô formula. (This is Karatzas-Shreve Ex 2.27 on page 299)

Problem 2: Let $\sigma\colon \mathbb{R} \to \mathbb{R}$ be a Borel function. Prove that solutions to the SDE

$$\mathrm{d}X_t = \sigma(X_t)\mathrm{d}B_t,$$

cannot explode in finite time. More precisely, if $\{X_t\colon t \in [0,\tau_\infty]\}$ is, for each $n \geqslant 1$, a weak solution to this SDE up to the stopping time

$$\tau_n := \inf\bigl\{t \geqslant 0\colon |X_t| \geqslant n\bigr\}$$

then $\tau_\infty := \lim_{n\to\infty} \tau_n$ (called the "explosion time") obeys $\tau_\infty = \infty$ a.s.

Problem 3: Let $\theta > 1$ and consider the SDE

$$\mathrm{d}X_t = (X_t^+)^\theta \mathrm{d}t + \mathrm{d}B_t$$

Prove that any solution started from $X_0 > 0$ explodes with positive probability. If you have the strength then show that, in fact, an explosion occurs with probability one.
Hint: Write the SDE in ODE form and compare it with an ODE that can be solved explicitly, assuming that B is suitably controlled via a stopping time.
