HW#6: due Fri 11/28/2025

The four problems below practice concepts from additive chaos theory and then hint at the connection of stochastic analysis to some standard PDEs that we have not had time to go over in class.

Problem 1: Let $t \geqslant 0$ and let $f \in L^2([0,t])$. Prove that then

$$\exp\left\{\int_0^t f(s)\mathrm{d}B_s - \frac{1}{2}\int_0^t f(s)^2\mathrm{d}s\right\} = 1 + \sum_{n\geqslant 1} I_t^{(n)}\big(1_{D_n} f^{\otimes n}\big) \quad \text{a.s.}$$

where $f^{\otimes n}\colon \mathbb{R}_+^n \to \mathbb{R}$ is defined by

$$f^{\otimes n}(t_1,\dots,t_n) := \prod_{i=1}^n f(t_i)$$

and $D_n := \{(t_1,\dots,t_n) \in \mathbb{R}^n\colon 0 \leqslant t_1 < \dots < t_n\}$. The object $I_t^{(n)}(h)$ is the iterated Itô integral of $h \in L^{2,\mathrm{loc}}(D_n)$. The sum above converges in L^2.

Hint: Denoting the exponential as M_t, derive an SDE for M.

Problem 2: Let $f, g \in L^2([0,t])$. Prove that

$$\left(\int_0^t f(s)\mathrm{d}B_s\right)\left(\int_0^t g(s)\mathrm{d}B_s\right) = I_t^{(2)}(1_{D_2} f \otimes g) + I_t^{(2)}(1_{D_2} g \otimes f) + \int_0^t f(s)g(s)\mathrm{d}s \quad \text{a.s.}$$

where the objects are as in the previous problem. (The formula expresses the "non-commutativity" of Brownian increments and the need to apply the rule $(\mathrm{d}B_t)^2 = \mathrm{d}t$ whenever they coincide. A similar formula exists for products of any number of Paley-Wiener integrals but the combinatorics of pairings of the increments makes it rather complicated.)

Problem 3: Prove the representation of a solution to Dirichlet problem for the Laplace equation: Let $D \subseteq \mathbb{R}^d$ be a non-empty bounded open set and $u \in C^2(D) \cap C(\overline{D})$ a function such that

$$\forall x \in D\colon \Delta u(x) = 0$$

Denoting by B the d-dimensional Brownian motion, write $\tau_D := \inf\{t \geqslant 0\colon B_t \notin D\}$ for the first time B exists the domain D. Prove that

$$\forall x \in D\colon\; P^x(\tau_D < \infty) = 1 \;\wedge\; u(x) = E^x\big(u(B_{\tau_D})\big)$$

where E^x is expectation with respect to the law P^x of B such that $P^x(B_0 = x) = 1$. (Note that this rewrites as $u(x) = \int P^x(B_{\tau_D} \in \mathrm{d}z)u(z)$, which for $P^x(B_{\tau_D} \in \cdot)$ absolutely continuous with respect to the surface measure on ∂D gives us a Poisson-kernel type representation of harmonic functions as an integral over its boundary values.) Then conclude the Maximum Principle:

$$\sup_{x\in D} |u(x)| \leqslant \sup_{z\in\partial D} |u(z)|$$

(It is a bit harder to show that $u(x) := E^x\big(g(B_{\tau_D})\big)$ actually defines a harmonic function with boundary values given by g, although even for g continuous that requires conditions on ∂D. By the Maximum Principle, such a solution is then unique.)

Hint: Check that $u(B_{\tau_D \wedge t})$ is a continuous martingale.

Problem 4: Prove the Feynman-Kac formula: Let $u \in C^{1,2}((0,\infty) \times \mathbb{R}^d) \cap C([0,\infty) \times \mathbb{R}^d)$ be bounded and such that, for some bounded continuous $V : \mathbb{R}^d \to \mathbb{R}$,

$$\forall t > 0\, \forall x \in \mathbb{R}^d : \quad \frac{\partial u}{\partial t}(t,x) = \tfrac{1}{2}\Delta u(t,x) + V(x)u(t,x)$$

Then

$$u(t,x) = E^x\left(u(0,B_t) \exp\Big\{\int_0^t V(B_s)\mathrm{d}s\Big\}\right)$$

where B is a standard Brownian motion that, under E^x, is started from $B_0 = x$.

Hint: Check that $M_s := u(t-s, B_s)\exp\{\int_0^s V(B_u)\mathrm{d}u\}$ is a continuous local martingale.
