HW#5: due Fri 11/21/2025

The problems below focus on continuous local martingales, their quadratic variation and stochastic integrals associated with these processes.

Problem 1: Let $\Omega := \mathbb{R}_+$ with $\mathcal{F} := \mathcal{B}(\mathbb{R}_+)$. Define $X_t \colon \Omega \to \mathbb{R}$ by $X_t(\omega) := 1_{\{t\}}(\omega)$. For each $t \geq 0$, let $\mathcal{F}_t$ be the least σ-algebra on Ω containing the sets $\{\{x\} \colon x \in \mathbb{R}_+\}$. Prove that $\{X_t \colon t \geq 0\}$ is jointly measurable, adapted to $\{\mathcal{F}_t\}_{t\geq 0}$ but not progressively measurable with respect to $\{\mathcal{F}_t\}_{t\geq 0}$.

Problem 2: Let M be a continuous local martingale. Do the following:

(1) If T is a stopping time, then $\{M_{T\wedge t} \colon t \geq 0\}$ is a continuous local martingale with $\forall t \geq 0 \colon \langle M_{T\wedge \cdot}\rangle_t = \langle M\rangle_{T\wedge t}$ a.s.

(2) M is a martingale if $\sup_{s\leq t} |M_s| \in L^1$ for all $t \geq 0$.

(3) If $M \geq 0$, then M is a supermartingale and we have

$$M \text{ martingale} \quad \Leftrightarrow \quad \forall t \geq 0 \colon E(M_t) = E(M_0)$$

Note: In (2), as much as it may seem reasonable, to get that M is a martingale it is not sufficient to assume that $\{M_s \colon s \leq t\}$ is uniformly integrable for all $t \geq 0$.

Problem 3: Let M and N be continuous local martingales and let $\langle M, N\rangle$ be their covariation process. Prove that there exists a measurable set $\Omega^\star$ with $P(\Omega^\star) = 1$ such that the following facts hold on $\Omega^\star$:

(1) $\forall t \geq 0 \colon \langle M, N\rangle_t \leq \sqrt{\langle M\rangle_t}\sqrt{\langle N\rangle_t}$

(2) $\forall t \geq s \geq 0 \colon$

$$\big|\langle M, N\rangle_t - \langle M, N\rangle_s\big| \leq \sqrt{\langle M\rangle_t - \langle M\rangle_s}\,\sqrt{\langle N\rangle_t - \langle N\rangle_s}$$

(3) $\forall t \geq s \geq 0 \colon$

$$\big|V_t^{(1)}(\langle M, N\rangle) - V_s^{(1)}(\langle M, N\rangle)\big| \leq \sqrt{\langle M\rangle_t - \langle M\rangle_s}\,\sqrt{\langle N\rangle_t - \langle N\rangle_s}$$

(4) (Kunita-Watanabe inequality) For all jointly measurable processes Y and $\widetilde{Y}$,

$$\int_0^t |Y_s \widetilde{Y}_s| \mathrm{d}V_t^{(1)}(\langle M, N\rangle) \leq \left(\int_0^t Y_s^2 \mathrm{d}\langle M\rangle_s\right)^{1/2} \left(\int_0^t \widetilde{Y}_s^2 \mathrm{d}\langle N\rangle_s\right)^{1/2}$$

holds for all $t \geq 0$ for which the integrals on the right are finite.

Hint: Use (3) to prove the inequality in (4) for simple processes and then take limits.

Problem 4: Assuming continuous versions of the stochastic integrals, for all continuous local martingales M, N, all $Y \in \mathcal{V}_M^{\text{loc}}$ and all $\widetilde{Y} \in \mathcal{V}_N^{\text{loc}}$,

$$\forall t \geq 0 \colon \left\langle \int_0^\cdot Y_s \mathrm{d}M_s, \int_0^\cdot \widetilde{Y}_s \mathrm{d}N_s \right\rangle_t = \int_0^t Y_s \widetilde{Y}_s \mathrm{d}\langle M, N\rangle_s$$

holds, with the Lebesgue-Stieltjes integral on the right finite, with probability one.

Hint: First prove this directly for Y and $\widetilde{Y}$ simple. Then invoke the Kunita-Watanabe inequality to perform extension to locally integrable processes.
