HW#3: due Mon 11/3/2025

The first two exercises are drawn from the discussion leading to the definition of stochastic (Itô) integral. The remaining three use the full-fledged form of the Itô integral.

Problem 1: Prove that for a.e. path B of the standard Brownian motion there is a sequence of partitions $\{\Pi_n\}$ with $\|\Pi_n\| \to 0$ such that

$$V_t^{(2)}(B, \Pi_n) \underset{n\to\infty}{\longrightarrow} \infty$$

This shows that the actual 2nd variation of the Brownian path is infinite a.s.

Hint: Consider using Khinchin's Law of the Iterated Logarithm.

Problem 2: Recall that a white noise on a finite measure space $(\mathscr{X}, \Sigma, \mu)$ is a Gaussian process $\{W(A) \colon A \in \Sigma\}$ that obeys $EW(A) = 0$ and $\mathrm{Cov}(W(A), W(B)) = \mu(A \cap B)$ for all $A, B \in \Sigma$. Endowing Σ with the pseudometric

$$A, B \mapsto \mu(A \triangle B)$$

(which becomes a metric under factoring out null sets), we can study continuity of sample paths $A \mapsto W(A)$. Prove that continuity — or more precisely, existence of a continuous version — fails already for the example $\mathscr{X} := \mathbb{N}$ and $\Sigma :=$ all subsets of $\mathbb{N}$. Specifically, characterize μ for which $A \mapsto W(A)$ is continuous away from a null event.

Hint: Observe that $\{\mathscr{X}(\{n\}) \colon n \in \mathbb{N}\}$ are independent centered normals.

Problem 3: Let $f \colon [0, t] \to \mathbb{R}$ be of bounded variation (meaning $\sup_\Pi V_t^{(1)}(f, \Pi) < \infty$). Prove that

$$\int_0^t f(s)\,\mathrm{d}B_s = f(t)B_t - f(0)B_0 - \int_0^t B_s\,\mathrm{d}f(s) \quad \text{a.s.}$$

where the latter is a Stieltjes integral. Then check that the same holds even if f is of bounded p-variation (meaning $\sup_\Pi V_t^{(p)}(f, \Pi) < \infty$) for some $p \in [1, 2)$. (This subsumes Øksendal ex. 3.1 page 37)

Problem 4: Prove that

$$\int_0^t B_s^2\,\mathrm{d}B_s = \frac{1}{3}B_t^3 - \int_0^t B_s\mathrm{d}s \quad \text{a.s.}$$

by invoking the previous problem as well as directly by suitably rearranging the associated Riemann-Stieltjes sums. (This subsumes Øksendal ex. 3.2 page 38.)

Problem 5: (Fubini for Itô) Let $f \colon [0, t] \to \mathbb{R}$ be continuous. Show that for any $Y \in \mathcal{V}$,

$$\int_0^t f(s)\Big(\int_0^s Y_u\,\mathrm{d}B_u\Big)\mathrm{d}s = \int_0^t \Big(\int_u^t f(s)\mathrm{d}s\Big) Y_u\,\mathrm{d}B_u \quad \text{a.s.}$$

Here $\mathcal{V}$ is the class of jointly measurable, square integrable processes adapted to a Brownian filtration (see definition in class).
