HW#2: due Mon 10/20/2025

Problem 1: (Gaussian Hilbert space) Let $\mathcal{H}$ be a Hilbert space over $\mathbb{R}$ with the scalar product denoted by $\langle\cdot,\cdot\rangle$. Show that there exists a process $\{X(f)\colon f\in\mathcal{H}\}$ such that for any $f_1,\dots,f_n\in\mathcal{H}$, the collection $(X(f_1),\dots,X(f_n))$ is multivariate normal with

$$EX(f)=0 \quad\text{and}\quad \operatorname{Cov}\bigl(X(f),X(g)\bigr)=\langle f,g\rangle$$

for all $f,g\in\mathcal{H}$. Then prove that $f\mapsto X(f)$ is linear and continuous as a map $\mathcal{H}\to L^2$.

Note: Under the circumstances when a.e. sample of $f\mapsto X(f)$ is a *continuous* linear functional on $\mathcal{H}$, the Riesz lemma yields the existence of an $\mathcal{H}$-valued random variable $\overline{X}$ such that $X(f)=\langle\overline{X},f\rangle$. The law of $\overline{X}$ is then a Gaussian measure on $\mathcal{H}$.

Problem 2: Let $\mathcal{H}$ be the completion of the set $\{f\in C^1[0,\infty)\colon f(0)=0, f'\in L^2([0,\infty))\}$ with respect to the norm-topology associated with the inner product

$$\langle f,g\rangle := \int_{[0,\infty)} \mathrm{d}x\, f'(x)g'(x)$$

and let X be the process defined in the previous problem. Do the following:

1. Show that $f_t(x):=\min\{x,t\}$ lies in $\mathcal{H}$ for each $t\ge 0$.
2. Prove that setting $B_t:=X(f_t)$ defines a process $\{B_t\colon t\ge 0\}$ whose finite dimensional distributions are those of the standard Brownian motion.
3. Check that, if $\{h_n\}_{n\ge1}$ is an orthonormal basis in $\mathcal{H}$, then $\{X(h_n)\colon n\ge\}$ are i.i.d. standard normals. Use this to conclude that, for all $t\ge0$ and B_t as in (2),

   $$B_t=\sum_{n\ge1}X(h_n)h_n(t)\quad\text{a.s.}$$

   with the sum converging in L^2. Prove that the sum also converges a.s.

Note: Part (3) gives a way to construct the standard Brownian motion as an random infinite series. For instance, letting $\{h_n\}_{n\ge1}$ be the antiderivatives of the Haar basis in $L^2([0,1])$ we recover the original Lévy construction. A key point there is that the series actually converges uniformly a.s. and thus yields directly a continuous process.

Problem 3: Let

$$C[0,\infty):=\bigl\{f\in\mathbb{R}^{[0,\infty)}\colon \text{ continuous}\bigr\}$$

and

$$\rho(f,g):=\sum_{n\ge1}2^{-n}\sup_{t,s\in[0,n]}\bigl|f(t)-g(t)\bigr|\wedge1$$

Prove that ρ is a metric such that $(C[0,\infty),\rho)$ is complete and separable. Denoting

$$U_k(f,a):=\Bigl\{g\in C[0,\infty)\colon \sup_{t\in[0,k]}|g(t)-f(t)|<a\Bigr\}.$$

then show that there exists $\{f_n\}_{n\ge1}\in C[0,\infty)$ such that every open set in $C[0,\infty)$ is a countable union of finite intersections of sets $\{U_k(f_n,a)\colon k,n\ge1,\ a\in\mathbb{Q}^+\}$.

Problem 4: Let $\mathcal{P} \subseteq \mathcal{B}(\mathbb{R})$ be a π-system such that $\sigma(\mathcal{P}) = \mathcal{B}(\mathbb{R})$. Prove that every probability measure μ on the Wiener space $(C[0,\infty), \mathcal{B}(C[0,\infty))$ is determined by its restriction to sets

$$\bigcap_{i=1}^{n} \{f \in C[0,\infty) \colon f(t_i) \in A_i\}$$

where $n \geq 1, t_1, \dots, t_n \in [0,\infty)$ and $A_1, \dots, A_n \in \mathcal{P}$. (In particular, every μ is determined by its finite-dimensional projections.)

Problem 5: Recall that $V_t^{(p)}(f,\Pi) := \sum_{i=1}^{n} |f(t_i) - f(t_{i-1})|^p$ is the p-variation of f for a partition Π of $[0,t]$ with partition points $0 = t_0 < t_1 < \dots < t_n = t$. Prove that, for $B = \{B_t \colon t \geq 0\}$ the standard Brownian motion and $f \in C^1(\mathbb{R})$,

$$V^{(2)}\big(f \circ B, \Pi\big) \underset{\|\Pi\| \to 0}{\xrightarrow{P}} \int_0^t f'(B_s)^2 \mathrm{d}s$$

Here $\|\Pi\| := \max_{i=1,\dots,n} |t_i - t_{i-1}|$ and the integral is in Riemann sense.
Hint: Use Taylor's theorem. First assume that f' is uniformly bounded.

Problem 6: Fix $\theta \in [0,1]$ and let $\{B_t \colon t \geq 0\}$ be a standard Brownian motion. For any $f \colon \mathbb{R} \to \mathbb{R}$, $t \geq 0$ and a partition $\Pi = \{0 = t_0 < t_1 < \dots < t_n = t\}$ of $[0,t]$ consider the Riemann-Stieltjes sum

$$I_t^{(\theta)}(f,\Pi) := \sum_{i=1}^{n} f(B_{(1-\theta)t_{i-1}+\theta t_i})(B_{t_i} - B_{t_{i-1}})$$

Assuming $f \in C^1(\mathbb{R})$ with f, f' bounded, prove that

$$I_t^{(\theta)}(f,\Pi_n) \underset{\|\Pi\| \to 0}{\xrightarrow{P}} \int_0^t f(B_s)\mathrm{d}B_s + \theta \int_0^t f'(B_s)\mathrm{d}s$$

Here the first integral is the Itô integral constructed in class.
Note: This shows the sensitivity of the Itô integral (which corresponds to $\theta := 0$, i.e., the left-endpoint) to the choice of the marked points in the partition. In particular, the Itô integral does not exist in Riemann-Stieltjes sense.
