HW#1: due Mon 10/6/2025

Problem 1: Let T be a set and let $\{X_t \colon t \in T\}$ be a family of random variables. Prove

$$\sigma(\{X_t \colon t \in T\}) = \bigcup_{\substack{S \subset T \\ \text{countable}}} \sigma(\{X_t \colon t \in S\})$$

Problem 2: Use the previous exercise to show that, given an $\mathbb{R}$-valued stochastic process $\{X_t \colon t \in [0,\infty)\}$ realized via coordinate projections on $(\mathbb{R}^{[0,\infty)}, \mathcal{B}(\mathbb{R})^{\otimes [0,\infty)})$, we have

$$\big\{t \mapsto X_t \text{ is continuous}\big\} \notin \sigma\big(X_t \colon t \in [0,\infty)\big)$$

Show also that for any $A \in \sigma(X_t \colon t \in [0,\infty))$,

$$A \subseteq \big\{t \mapsto X_t \text{ is continuous}\big\} \;\Rightarrow\; A = \emptyset$$

Problem 3: ØKSENDAL EX 2.8, PAGE 17

Problem 4: Prove that conditions 2-3 in our definition of standard Brownian motion — namely, the independence of increments along with $B_t - B_s = \mathcal{N}(0, |t-s|)$ — are equivalent to

$$\{B_t \colon t \geq 0\} \text{ is Gaussian with } \forall t, s \geq 0\colon\; EB_t = 0 \;\wedge\; E(B_t B_s) = t \wedge s$$

Here we recall that random variables $\{X_t \colon t \in T\}$ are called Gaussian (a.k.a. multivariate normal) if for any $\lambda \colon T \to \mathbb{R}$ with $\{t \in T \colon \lambda(t) \neq 0\}$ finite, the (effectively finite) sum $\sum_{t \in T} \lambda(t) X_t$ is (univariate) normal.

Problem 5: Let $\{B_t \colon t \geq 0\}$ be the d-dimensional Brownian motion — namely, an $\mathbb{R}^d$-valued stochastic process whose Cartesian components are independent standard Brownian motions — and let

$$L(A) = \int_0^\infty 1_{\{B_s \in A\}} \mathrm{d}s$$

be the total time spent in a Borel set A. Check that this is a well-defined random variable. Then prove that, for A a Lebesgue null set we have $L(A) = 0$ a.s. (This is Øksendal ex. 2.14 page 19)

Problem 6: Let $\{B_t \colon t \geq 0\}$ be the d-dimensional Brownian motion and let U be an orthogonal matrix. Show that also UB_t is a d-dimensional Brownian motion. (This is Øksendal ex. 2.15 page 19)
