Itô integral

GOAL: Define $\int_0^t Y_s \, dB_s$ for good amount of Y's.

Def A stoch. process on $(\Omega, \mathcal{F}, P)$ with filtration $\{\mathcal{F}_t\}_{t \in \mathbb{R}_+}$ is simple if $\exists n \in \mathbb{N}$ $\exists 0 = t_0 < t_1 < \dots < t_n$ $\exists Z_0, \dots, Z_n$ RVs s.t.

1) $\forall i = 0 \dots n$: $Z_i \in L^\infty \wedge Z_i$ is $\mathcal{F}_{t_{i-1}}$-meas. $(t_{-1} := 0)$

2) $\forall t \geq 0$: $Y_t = Z_0 1_{\{0\}}(t) + \sum_{i=1}^n Z_i \, 1_{(t_{i-1}, t_i]}(t)$

Notation: $\mathcal{V}_0 := \{Y : \text{simple}\}$ ("Step" process)

Lemma For $Y \in \mathcal{V}_0$ with above representation, let

$$\int_0^t Y_s \, dB_s := \sum_{i=1}^n Z_i \left(B_{t_i \wedge t} - B_{t_{i-1} \wedge t} \right)$$

Then this does NOT depend on representation of Y.

Pf: skipped

Lemma (Itô isometry)

$$\forall Y \in \mathcal{V}_0\ \forall t \geq 0: \ E\left[\left(\int_0^t Y_s \, dB_s\right)^2\right] = E \int_0^t Y_s^2 \, ds$$

Pf:
$$\text{LHS} = \sum_{i,j=1}^{n} E\left(Z_i Z_j \left(B_{t_i \wedge t} - B_{t_{i-1}\wedge t}\right)\left(B_{t_j \wedge t} - B_{t_{j-1}\wedge t}\right)\right)$$
$$= \sum_{i=1}^{n} E\left(Z_i^2 \left(B_{t_i \wedge t} - B_{t_{i-1}\wedge t}\right)^2\right)$$
$$= \sum_{i=1}^{n} E\left(Z_i^2 \left(t_i \wedge t - t_{i-1}\wedge t\right)\right) = \text{RHS} \quad \boxtimes$$

Need a good L^2 space:

Def Let $\mathcal{V}$ be set of all stoch processes Y s.t.

1) $(\omega, t) \longmapsto Y_t(\omega)$ is jointly measurable (with respect to $\mathcal{F} \otimes \mathcal{B}(\mathbb{R}_+)$)
2) Y is adapted meaning $\forall t \in \mathbb{R}_+$: Y_t is $\mathcal{F}_t$-measurable
3) $\forall t \in \mathbb{R}_+$: $\|Y\|_{L^2([0,t]\times\Omega)} := \left[E\left(\int_0^t Y_s^2 \, ds\right)\right]^{1/2} < \infty.$

Lemma 1) $\mathcal{V}_0 \subseteq \mathcal{V}$

2) $\mathcal{V}$ is linear vector space under pointwise addition and scalar multiplication

3) $[\![Y]\!] := \sum_{n \geq 1} 2^{-n} \min\{1, \|Y\|_{L^2([0,n]\times\Omega)}\}$

defines a pseudometric via $Y, \tilde{Y} \mapsto [\![Y - \tilde{Y}]\!]$

s.t. $[\![Y - \tilde{Y}]\!] = 0 \Rightarrow \mathrm{Leb}\otimes P\left((t,\omega) \in \mathbb{R}_+ \times \Omega : Y_t(\omega) \neq \tilde{Y}_t(\omega) \right) = 0$

In particular, the space of equivalence classes $\{ \{\tilde{Y} \in \mathcal{V} : [\![\tilde{Y} - Y]\!] = 0\} : Y \in \mathcal{V} \}$ is a metric space.

Corollary Set $\overline{\mathcal{V}_0}^{[\![\cdot]\!]} :=$ closure of $\mathcal{V}_0$ in $\mathcal{V}$.

Then $\forall t \geq 0$: $Y \to \int_0^t Y \, dB_s$ extends uniquely to $\overline{\mathcal{V}_0}^{[\![\cdot]\!]}$

and obeys Itô isometry. Moreover, $\forall \alpha, \beta \in \mathbb{R}$ $\forall Y, \tilde{Y} \in \overline{\mathcal{V}_0}^{[\![\cdot]\!]}$

$$\int_0^t (\alpha Y_s + \beta \tilde{Y}_s) \, dB_s \overset{\text{a.s.}}{=} \alpha \int_0^t Y_s \, dB_s + \beta \int_0^t \tilde{Y}_s \, dB_s$$

(depends on $\alpha, \beta, Y, \tilde{Y}$ and t)

Key question: What is $\overline{\mathcal{V}_0}^{[\cdot]}$?

Lemma Suppose $Y \in \mathcal{V}$ has left-continuous paths. Then $Y \in \overline{\mathcal{V}_0}^{[\cdot]}$.

Pf Suppose first that Y is bounded ($\sup_{t\geq 0} \|Y_t\|_\infty < \infty$).

For $n \in \mathbb{N}$ set:

$$Y_t^{(n)} := Y_0 \mathbb{1}_{\{0\}}(t) + \sum_{k=0}^{4^n} Y_{k2^{-n}} \mathbb{1}_{(k2^{-n}, (k+1)2^{-n}]}(t)$$

Left continuity $\Rightarrow \forall t \geq 0$: $Y_t^{(n)} \to Y_t$.

$$\| Y^{(n)} - Y \|^2_{L^2([0,t]\times\Omega)} = E\int_0^t |Y_s^{(n)} - Y_s|^2 ds \xrightarrow[n\to\infty]{BCT} 0.$$

Then $[\![Y^{(n)} - Y]\!] \xrightarrow[n\to\infty]{} 0$ so $Y \in \overline{\mathcal{V}_0}^{[\cdot]}$.

Now take Y LC only and set $Y_t^{(M)} := Y_t \wedge M \vee (-M)$.

Then $Y^{(M)}$ is bdd & LC so $Y^{(M)} \in \overline{\mathcal{V}_0}^{[\cdot]}$.

$$\| Y^{(M)} - Y \|^2_{L^2([0,t]\times\Omega)} = E\int_0^t Y_s^2 \mathbb{1}_{\{|Y_s| > M\}} ds \xrightarrow[M\to\infty]{DCT} 0$$

So $Y \in \overline{\mathcal{V}_0}^{[\cdot]}$. $\boxtimes$

Corollary: Any $Y \in \mathcal{V}$ with continuous paths lies in $\mathcal{V}_0[0,T]$.

Q: Calculate $\int_0^t Y_s \, dB_s$ explicitly?

Ex: $\int_0^t dB_s = B_t - B_0$

$$\int_0^t B_s \, dB_s = \frac{1}{2}(B_t^2 - t)$$

$$= \frac{1}{2}B_t^2 - \frac{1}{2}B_0^2 - \frac{1}{2}\int_0^t 1 \, ds$$

Itô formula:

$$f(B_t) = f(B_0) + \int_0^t f'(B_s)\, dB_s + \frac{1}{2}\int_0^t f''(B_s)\, ds$$

replaces FTC!

$$\int_0^t f(B_s)\, dB_s = F(B_t) - F(B_0) - \frac{1}{2}\int_0^t f'(B_s)\, ds$$

where F = antiderivative of f.

IBP:

$$\int_0^t f(s)\, dB_s = f(t) B_t - f(0) B_0 - \int_0^t B_s\, df(s)$$

Suffices if f is $BV([0,t])$

$$\int_0^t s\, dB_s = \underline{t B_t} - \underline{\int_0^t B_s\, ds}$$