Random "measures" + stoch integral

Def (PPP) Given a finite meas. space $(\mathcal{X}, \Sigma, \mu)$ the Poisson Point Process with intensity μ — notation $PPP(\mu)$ — is $\{N(A) : A \in \Sigma\}$ s.t.

1) $\forall A \in \Sigma : \; N(A) \overset{law}{=} \text{Poisson}(\mu(A))$

2) $\forall \{A_i\}_{i\in\mathbb{N}} \in \Sigma^{\mathbb{N}}$: disjoint $\Rightarrow N\left(\bigcup_{i\in\mathbb{N}} A_i\right) = \sum_{i=0}^{\infty} N(A_i)$ a.s.

Lemma There exists a random measure N on $(\mathcal{X}, \Sigma)$ s.t. $\{N(A) : A \in \Sigma\}$ is $PPP(\mu)$.

Pf: Let $K = \text{Poisson}(\mu(\mathcal{X}))$, $\{X_i\}_{i\in\mathbb{N}}$ iid with law $\mu(\cdot)/\mu(\mathcal{X})$, independent of K. Now set

$$N(A) := \sum_{i=1}^{K} \delta_{X_i}(A) = \sum_{i=1}^{K} 1_A(X_i)$$

$$P(N(A)=k) = \sum_{n=0}^{\infty} P(N(\mathcal{X})=n)\, P\Big(\sum_{i=1}^{n} 1_A(X_i)=k\Big)$$

$$= \sum_{n=k}^{\infty} \frac{\mu(\mathcal{X})^n}{n!} e^{-\mu(\mathcal{X})} \binom{n}{k} \left(\frac{\mu(A)}{\mu(\mathcal{X})}\right)^k \left(1-\frac{\mu(A)}{\mu(\mathcal{X})}\right)^{n-k}$$

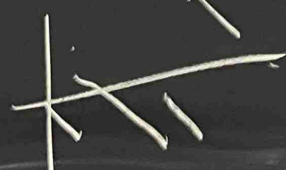
$$= \frac{\mu(A)^k}{k!} e^{-\mu(A)} \underbrace{\sum_{l=0}^{\infty} \frac{1}{l!}\mu(A^c)^l e^{-\mu(A^c)}}_{=1}$$

independence checked similarly. $\boxtimes$

<u>Remarks:</u>

- construction extends to σ-finite spaces
- For $\mu = \text{Leb}$ on $[0,\infty)$ we set

  $N_t := N([0,t])$. Then $\{N_t : t \in \mathbb{R}_+\}$ is (ordinary) Poisson process.
- zero mean is achieved by compensated process: $\hat{N}_t := N_t - t$



Def Given a finite meas. space $(\mathcal{X}, \Sigma, \mu)$, a white noise associated with μ is a stoch process $\{W(A) : A \in \Sigma\}$ s.t.

1) $\forall A \in \Sigma : \quad W(A) = \mathcal{N}(0, \mu(A))$

2) $\forall \{A_i\}_{i \in \mathbb{N}} \in \Sigma^{\mathbb{N}}$ disjoint $\Longrightarrow$ $\begin{cases} \{W(A_i)\}_{i \in \mathbb{N}} \text{ are independent} \\ \text{and } W\left(\bigcup_{i \in \mathbb{N}} A_i\right) = \sum_{i=0}^{\infty} W(A_i) \text{ a.s.} \end{cases}$

Note: By indep + $\sum_{i=0}^{\infty} \mu(A_i) < \infty \Rightarrow \lim_{n \to \infty} \sum_{i=0}^{n} W(A_i)$ exists a.s.

Lemma A white noise exists

Pf: $A, B \in \Sigma$

$A = A \cap B \cup A \setminus B$

$B = A \cap B \cup B \setminus A$

$\mathrm{Cov}(W(A) W(B)) = \mathrm{Var}(W(A \cap B)) = \mu(A \cap B)$

Let $\{ W(A) : A \in \Sigma \}$ be Gaussian with
$\mathbb{E} W(A) = 0$ and $\mathrm{Cov}(W(A) W(B)) = \mu(A \cap B)$

check that $A, B \to \mu(A \cap B)$ is pos. semidef.

Then if $\{A_i\}_{i \in \mathbb{N}}$ are disjoint, then $\{W(A_i)\}_{i \in \mathbb{N}}$ are independent.

Moreover, letting $A := \bigcup_{i \in \mathbb{N}} A_i$, we get

$$\mathbb{E}\left(\left| W(A) - \sum_{i=0}^{n} W(A_i) \right|^2 \right) = \mu(A) - \sum_{i=0}^{n} \mu(A_i) \xrightarrow[n \to \infty]{} 0$$

So $$W(A) = \sum_{i=0}^{\infty} W(A_i) \qquad \boxtimes$$

Q: Is $A \mapsto W(A)$ a signed measure?

Q: Is $A \mapsto W(A)$ continuous wrt. $A, B \mapsto \mu(A \Delta B)$ metric.

A: No unless $(\mathcal{X}, \Sigma)$ is "trivial".

Notwithstanding, we can "integrate" wrt W:

Give $f := \sum_{i=1}^{n} a_i 1_{A_i}$ set

$$\int f \, dW := \sum_{i=1}^{n} a_i W(A_i)$$

<u>Lemma</u> $\int f \, dW$ does not depend on representation of f.
(modulo changes in null sets with respect to P)

<u>Pf</u>: Finite additivity a.s.

<u>Lemma</u> $\forall f$ simple:

$$E\left(\left(\int f \, dW\right)^2\right) = \int f^2 \, d\mu. \quad (*)$$

<u>Pf</u>: $\text{LHS} = E\left(\left(\sum_{i=1}^{n} a_i W(A_i)\right)^2\right) \underset{\substack{\text{WLOG} \\ \{A_i\} \text{ disjoint}}}{=} \sum_{i,j} a_i a_j E(W(A_i) W(A_j))$

$$= \sum_{i=1}^{n} a_i^2 \mu(A_i) = \text{RHS} \quad \boxtimes$$

Think of $(*)$ as isometry of $L^2(\mathcal{X},\Sigma,\mu) \longrightarrow L^2(\Omega,\mathcal{F},P)$

since simple functions are dense in $L^2(\mathcal{X},\Sigma,\mu)$ we have defined $\int f\,dW$ for all $f \in L^2(\mathcal{X},\Sigma,\mu)$ s.t. $(*)$ holds.

We call $\int f\,dW$ the Paley-Wiener integral.

Itô integral

GOAL: Define $\int_0^t Y_s\,dB_s$ for sufficiently many $\{Y_s : s \in \mathbb{R}_+\}$

WANT (various reasons): Y_t depends only on $\{B_s : s \le t\}$.

Def Given a SBM $\{B_t : t \in \mathbb{R}_+\}$ on $(\Omega, \mathcal{F}, P)$, a family $\{\mathcal{F}_t\}_{t\in\mathbb{R}_+}$ is a <u>Brownian filtration</u> if

1) $\forall\, 0 \le s < t$: $\mathcal{F}_s \subseteq \mathcal{F}_t \subseteq \mathcal{F}$.
2) $\forall\, t \ge 0$: B_t is $\mathcal{F}_t$-measurable (B <u>adapted</u>)
3) $\forall\, t \ge 0$: $\sigma(B_{s+t} - B_t : s \ge 0) \perp\!\!\!\perp \mathcal{F}_t$. (B is Markovian)

<u>Next time</u>

- simple process
- integral for these
- extension via Itô isometry