Motivation for stochastic integral

Lemma For $f \in C(\mathbb{R})$, $\Pi = \{0 = t_0 < \dots < t_n = t\}$ set

$$I_t(f,\Pi) := \sum_{i=1}^{n} f(B_{t_{i-1}})\left(B_{t_i} - B_{t_{i-1}}\right)$$

Thm $\forall f \in C_b^2(\mathbb{R})$ $\exists\, I_t(f) = \text{RV}$ s.t.

1) $\|\Pi_n\| \to 0 \implies I_t(f,\Pi_n) \xrightarrow[n\to\infty]{P} I_t(f)$

2) $f(B_t) = f(B_0) + I_t(f') + \frac{1}{2}\int_0^t f''(B_s)\,ds$ a.s.

Notation: $\int_0^t f(B_s)\,dB_s =: I_t(f)$

Pf: $$f(B_t) - f(B_0) = I_t(f',\Pi) + \frac{1}{2}\sum_{i=1}^{n} f''(B_{t_{i-1}})(t_i - t_{i-1}) + L(f,\Pi)$$

where $$L(f,\Pi) = \frac{1}{2}\sum_{i=1}^{n} f''(B_{t_{i-1}})\left[(B_{t_i} - B_{t_{i-1}})^2 - (t_i - t_{i-1})\right] \leftarrow L_1(f,\Pi)$$

$$+ \sum_{i=1}^{n}\left(\int_0^1 \underbrace{\left[f''(\theta B_{t_i} + (1-\theta)B_{t_{i-1}}) - f''(B_{t_{i-1}})\right]}_{\le\, \operatorname{osc}_{f''\circ B}([0,t],\|\Pi\|) \xrightarrow[\|\Pi\|\to 0]{} 0}(1-\theta)\,d\theta\right)\left(B_{t_i} - B_{t_{i-1}}\right)^2$$

$\nwarrow L_2(f,\Pi)$

$$|L_2(f, \Pi)| \le \operatorname{osc}_{f'' \circ B}([0,t], \|\Pi\|)\, V_t^{(2)}(B, \Pi)$$

so $L_2(f, \|\Pi_n\|) \xrightarrow[n \to \infty]{P} 0$.

$$E(|L_1(f, \Pi)|^2) = \sum_{i=1}^{n} E(f''(B_{t_{i-1}})^2)\, \mathrm{Var}(|B_{t_i} - B_{t_{i-1}}|^2)$$

$$= \mathrm{Var}(N(0,1)^2) \sum_{i=1}^{n} E(f''(B_{t_{i-1}})^2)\, |t_i - t_{i-1}|^2$$

$$\le C t \|\Pi\| \xrightarrow[\|\Pi\| \to 0]{} 0$$

$\boxtimes$

Note For limit of $I_t(f, \Pi)$ is suffici^t $f \in C_b(\mathbb{R})$.

Lemma Let $f \in C_b(\mathbb{R})$. Then $\exists\, I_t(f)$ RV s.t.

$$\|\Pi_n\| \to 0 \Rightarrow I_t(f, \Pi_n) \xrightarrow[n \to \infty]{P} I_t(f)$$

Pf By Lemma: $I_t(f, \Pi_n)$ converges when $f' \in C_b(\mathbb{R})$

for $f \in C_b(\mathbb{R})$, need to aproximate. Pick $\varepsilon > 0$ let $M > 0$ be s.t. $P\left(\sup_{s \le t} |B_s| > M\right) < \varepsilon$.

Now pick $f_\varepsilon \in C_b^1(\mathbb{R}) \cap C_b(\mathbb{R})$:

$$\sup_{|x| \le M} |f(x) - f_\varepsilon(x)| \le \sqrt{\varepsilon}\, \|f\|, \quad \|f_\varepsilon\| \le 2\|f\|$$

sup norm on $\mathbb{R}$

$$\mathbb{E}\left(|I_t(f, \Pi) - I_t(f_\varepsilon, \Pi)|^2\right)$$
$$= E\left(\left|\sum_{i=1}^n (f - f_\varepsilon)(B_{t_{i-1}})(B_{t_i} - B_{t_{i-1}})\right|^2\right)$$
$$= E\left(\sum_{i=1}^n (f - f_\varepsilon)(B_{t_{i-1}})^2 |t_i - t_{i-1}|\right)$$
$$\le \|f - f_\varepsilon\|^2 t\, \varepsilon + \varepsilon \|f\|^2 t \le C t \varepsilon$$

Markov/Chebyshev:

$$P\left(| I_t(f,\Pi) - I_t(f_\varepsilon,\Pi) | > \delta \right) \le \frac{1}{\delta^2} C t \varepsilon \qquad (*)$$

This argument is not quite right due to a typo. See notes for an alternative.

Let $\{\Pi_n\}$ be s.t. $\|\Pi_n\| \to 0$. Set $\delta_n := 2^{-n}$, $\varepsilon_n = 8^{-n}$

Then $$P\left(| I_t(f_{\varepsilon_n},\Pi_n) - I_t(f_{\varepsilon_{n+1}},\Pi_n) | > 2^{-n+1} \right) \le 2 \cdot C t\, 2^{-n}$$

n+1 needed here

Borel-Cantelli: $I_t(f) := \lim_{n\to\infty} I_t(f_{\varepsilon_n},\Pi_n)$ exists a.s.

$(*)$ implies $I_t(f,\Pi_n) \xrightarrow[n\to\infty]{P} I_t(f)$. Showing $I_t(f_\varepsilon) \xrightarrow[\varepsilon\to 0]{P} I_t(f)$ using same argument finishes the proof. $\boxtimes$

<u>Note</u> $I_t(f)$ is NOT Stieltjes integral.

E.g. $f(x) = x$. Then

$$\sum_{i=1}^{n} B_{t_i}(B_{t_i} - B_{t_{i-1}}) - \sum_{i=1}^{n} B_{t_{i-1}}(B_{t_i} - B_{t_{i-1}})$$

$$= V_t^{(2)}(B,\Pi)$$

Lemma Let $f \in C_b^1(\mathbb{R})$. Then $\forall \theta \in [0,1]$, Π = partition set

$$I_t^{(\theta)}(f, \Pi) := \sum_{i=1}^{n} f(B_{\theta t_i + (1-\theta) t_{i-1}})(B_{t_i} - B_{t_{i-1}})$$

Then

$$\|\Pi_n\| \to 0 \implies I_t^{(\theta)}(f, \Pi_n) \longrightarrow \int_0^t f(B_s)\,dB_s + \theta \int_0^t f'(B_s)\,ds$$

Pf: HW3.

When $\theta := 1/2$ we call RHS the Stratonovich integral

$$\int_0^t f(B_s) \circ dB_s := \int_0^t f(B_s)\,dB_s + \frac{1}{2}\int_0^t f'(B_s)\,ds$$

Then $f(B_t) = f(B_0) + \int_0^t f'(B_s) \circ dB_s$.

ODE-type motivation for stoch integral

classical mech: $$\frac{dX_t}{dt} = a(t, X_t) \qquad \text{(eq. of motion)}$$

adding noise: $$\frac{dX_t}{dt} = a(t, X_t) + W_t$$

noise

WANT

1) $E W_t = 0$

2) $\forall s<t: \; W_t \perp\!\!\!\perp W_s$

3) $t \longmapsto W_t$ measurable

no such W (non-trivial) exists !

Idea: Write in integral form: $$X_t = X_0 + \int_0^t a(s, X_s)\, ds + \bar{W}([0,t])$$

where $\bar{W}$ is a "signed measure",