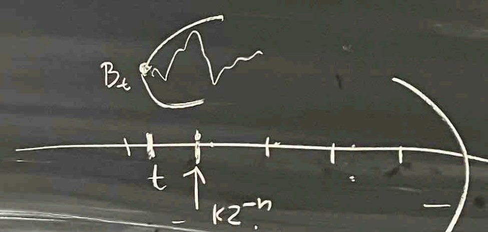
Thm (Paley-Wiener-Zygmund) Let $\{B_t : t\in[0,\infty)\}$ be standard Brownian motion defined on some $(\Omega,\mathcal{F},P)$. Then $\exists\, \Omega^*\in\mathcal{F}$ with $P(\Omega^*)=1$ s.t.

$$\forall \gamma\in(1/2,1]\ \forall t\geq 0:\ \limsup_{s\downarrow t}\frac{|B_t-B_s|}{|t-s|^\gamma}=\infty \quad \text{on } \Omega^*$$

In particular, a.e. path of B is nowhere differentiable.



Pf: Fix $\gamma\in(1/2,1]$, let $l\in\mathbb{N}$ be s.t. $l(\gamma-1/2)>1$.

Assume $\exists\, t\in[0,1/2]$ s.t. $\limsup_{s\downarrow t}\frac{|B_s-B_t|}{|s-t|^\gamma}<\infty$.

So $\exists\, \delta_0>0\ \exists\, C>0\ \forall \delta\in(0,\delta_0):\ |B_{t+\delta}-B_t|\leq C\delta^\gamma$

Let $n_0=\inf\{n\in\mathbb{N} : (l+2)2^{-n}<\delta_0\}$

Then for $n\geq n_0$ set $k:=\lceil 2^{+n}t\rceil$ and estimate:

for $j = 1, \dots, \ell$:

$$|B_{(k+j)2^{-n}} - B_{(k+j-1)2^{-n}}| \le |B_{(k+j)2^{-n}} - B_t| + |B_{(k+j-1)2^{-n}} - B_t|$$

$$\le C|(k+j)2^{-n} - t|^\gamma + C|(k+j-1)2^{-n} - t|^\gamma$$

So we get

$$\le 2 \cdot C[(\ell+1)2^{-n}]^\gamma = \underbrace{2 \cdot C(\ell+1)^\gamma}_{\le} 2^{-n\gamma}$$

$$\left\{ \exists t \in [0, 1/2] : \limsup_{s \to t} \frac{|B_s - B_t|}{|t-s|^\gamma} < \infty \right\} \subseteq \bigcup_{m \ge 1} \bigcup_{n_0 \ge 1} \bigcap_{n \ge n_0} \underbrace{\bigcup_{k=1}^{2^n} \bigcap_{j=1}^{\ell} \left\{ |B_{(k+j)2^{-n}} - B_{(k+j-1)2^{-n}}| \le m 2^{-n\gamma} \right\}}_{A_{n,m}}$$

Now $P(|B_{(k+j)2^{-n}} - B_{(k+j-1)2^{-n}}| \le m 2^{-n\gamma})$

$$= P(|N(0, 2^{-n})| \le m 2^{-n\gamma})$$

$$= P(|N(0,1)| \le m 2^{-n(\gamma - 1/2)}) \le 2 \cdot m 2^{-n(\gamma - 1/2)}$$

So $P(A_{n,m}) \le 2^n \cdot [2m 2^{-n(\gamma - 1/2)}]^\ell = (2m)^\ell 2^{-n[\ell(\gamma - 1/2) - 1]}$ $\boxed{\times}$

BC: $P(A_{n,m} \text{ i.o.}(n)) = 0$, so union over m trivial.

Thm (Law of Iterated log, Khinchin '33)

$$\forall t \geq 0: \quad \limsup_{s \downarrow t} \frac{|B_t - B_s|}{\sqrt{|t-s| \log\log \frac{1}{|t-s|}}} = \sqrt{2} \quad \text{a.s.} \ (t)$$

Cor

$$\limsup_{t \to \infty} \frac{|B_t|}{\sqrt{t \log\log t}} = \sqrt{2} \quad \text{a.s.}$$

Thm (Lévy modulus of continuity '37)

$$\limsup_{\delta \downarrow 0} \sup_{\substack{0 \leq s < t < \infty \\ |s-t| < \delta}} \frac{|B_t - B_s|}{\sqrt{\delta \log 1/\delta}} = \sqrt{2} \quad \text{a.s.}$$

fast points

Thm (Dvoretzky '63, Davis '83)

$$\inf_{t \geq 0} \limsup_{s \downarrow t} \frac{|B_s - B_t|}{\sqrt{|s-t|}} = 1$$

slow point

Quadratic variation

Given $\Pi = \{0 = t_0 < \dots < t_n = t\}$ a partition of $[0,t]$ and $p > 0$, we let $V_t^{(p)}(f, \Pi) := \sum_{i=1}^{n} |f(t_i) - f(t_{i-1})|^p$

Q: What's $V_t^{(p)}(B, \Pi)$?

$$E\, V_t^{(p)}(B, \Pi) = \sum_{i=1}^{n} E\left(|B_{t_i} - B_{t_{i-1}}|^p\right) = E\left(|N(0,1)|^p\right) \underbrace{\sum_{i=1}^{n} |t_i - t_{i-1}|^{\frac{p}{2}}}_{\leq \|\Pi\|^{\frac{p}{2}-1} t}$$

This $\xrightarrow[\|\Pi\| \to 0]{} 0$ for $p > 2$ and equals t for $p = 2$!

$$\mathrm{Var}\left(V_t^{(2)}(B, \Pi)\right) = \sum_{i=1}^{n} \mathrm{Var}\left(|B_{t_i} - B_{t_{i-1}}|^2\right)$$

$$= \mathrm{Var}\left(|N(0,1)|^2\right) \underbrace{\sum_{i=1}^{n} |t_i - t_{i-1}|^2}_{\leq \|\Pi\| t}$$

Lemma $\forall t \geq 0$:

1) $V_t^{(2)}(B, \Pi) \xrightarrow[\|\Pi\| \to 0]{P} t$

2) if $\sum_{n \geq 1} \|\Pi_n\| < \infty \Rightarrow V_t^{(2)}(B, \Pi_n) \xrightarrow[n \to \infty]{} t$ a.s.

3) if $\{\Pi_n\}_{n \geq 1}$ are nested, then

$\|\Pi_n\| \to 0 \Rightarrow V_t^{(2)}(B, \Pi_n) \xrightarrow[n \to \infty]{} t$ a.s.

Pf 1) proved above

2) $\forall \varepsilon > 0$: $P(|V_t^{(2)}(B, \Pi) - t| > \varepsilon)$

$\leq \frac{1}{\varepsilon^2} Var(V_t^{(2)}(B, \Pi)) \leq \frac{3t}{\varepsilon} \|\Pi\|$

Now apply Borel-Cantelli

3) based on martingale convergence, see notes. $\boxtimes$

Discovering Itô integral/formula

Let $f \in C^2(\mathbb{R})$ and Π a partition of $[0,t]$.

$$f(B_t) - f(B_0) = \sum_{i=1}^{n} \left[f(B_{t_i}) - f(B_{t_{i-1}}) \right]$$

$$= \sum_{i=1}^{n} f'(B_{t_{i-1}})(B_{t_i} - B_{t_{i-1}}) + \frac{1}{2}\sum_{i=1}^{n} f''(B_{t_{i-1}})(B_{t_i} - B_{t_{i-1}})^2$$

$$+ \sum_{i=1}^{n} \left(\int_0^1 \underbrace{\left[f''(\theta B_{t_{i-1}} + (1-\theta) B_{t_i}) - f''(B_{t_{i-1}}) \right]}_{\le \operatorname{osc}_{f''}(B[0,t],\, \|\Pi\|)} (1-\theta)\, d\theta \right) (B_{t_i} - B_{t_{i-1}})^2$$

Lemma For $f \in C(\mathbb{R})$ set $I_t(f,\Pi) := \sum_{i=1}^{n} f(B_{t_{i-1}})(B_{t_i} - B_{t_{i-1}})$

Then $\forall f \in C^2(\mathbb{R})$ $\exists\, I_t(f')$ RV s.t.

$$\|\Pi_n\| \to 0 \quad \Rightarrow \quad I_t(f', \Pi_n) \to I_t(f')$$

and

$$f(B_t) = f(B_0) + \underset{\substack{\|\ \text{notation} \\ \int_0^t f'(B_s)\, dB_s}}{I_t(f')} + \frac{1}{2}\int_0^t f''(B_s)\, ds$$