Uniqueness of SBM + path properties

Uniqueness cannot be gleaned from FDD's.

Idea: Think of SBM as a random continuous function.

Lemma Set $C[0,\infty) := \{ f \in \mathbb{R}^{[0,\infty)} : \text{continuous} \}$

$$\rho(f,g) := \sum_{n \geq 1} 2^{-n} \min\{ 1, \sup_{t \in [0,n]} |f(t) - g(t)| \}$$

Then $(C[0,\infty), \rho)$ is a complete & separable metric space.

So $(C[0,\infty), \mathcal{B}(C[0,\infty)))$ is standard Borel.

Lemma For all stoch processes $\{X_t : t \in [0,\infty)\}$ on $(\Omega, \mathcal{F}, P)$, $\mathbb{R}$-valued and such that $\forall \omega \in \Omega$: $t \mapsto X_t(\omega)$ is continuous, the map

$$\omega \mapsto \underbrace{(t \mapsto X_t(\omega))}_{=: X} \text{ is a } C[0,\infty)\text{-valued RV.}$$

Pf: NTS: $\forall A \in \mathcal{B}(C[0,\infty))$: $\{X \in A\} \in \mathcal{F}$.

Fact: $\mathcal{G} = \{ A \subseteq C[0,\infty) : \{X \in A\} \in \mathcal{F} \}$ is σ-alg.

So suffices to show that $O \in \mathcal{G}$ for all open $O \subseteq C[0,\infty)$.

Let $U_k(f,r) := \{ g \in C[0,\infty) : \sup_{t\in[0,k]} |f(t)-g(t)| < r \}$

Then separability implies $\exists \{f_n\}_{n\in\mathbb{N}}$ in $C[0,\infty)$ s.t.

$\forall O \subseteq C[0,\infty)$ open

$$O = \bigcup_{n\geq 1}\ \bigcup_{k\geq 1}\ \bigcup_{\substack{r\in\mathbb{Q}_+ \\ U_k(f_n,r)\subseteq O}} U_k(f_n,r)$$

<u>NTS</u>: $\forall f\in C[0,\infty)\ \forall r>0\ \forall k\geq 1 :\ U_k(f,r)\in\mathcal{G}$.

This follows for

$$\{X\in U_k(f,r)\} = \bigcup_{\substack{a\in\mathbb{Q}_+ \\ a<r}}\ \bigcap_{\substack{t\in\mathbb{Q}_+ \\ t\leq k}} \underbrace{\{|X_t - f(t)| \leq a\}}_{\in\mathcal{F}}$$

In fact, we showed $\{X\in O\}\in\sigma(X_t : t\in[0,\infty))$. $\boxtimes$

Corollary Every $\mathbb{R}$-valued stoch. process $\{X_t : t \in [0,\infty)\}$ defined on $(\Omega, \mathcal{F}, P)$ induces a prob. measure P^X on $\underbrace{(C[0,\infty), \mathcal{B}(C[0,\infty)))}$ via

$$P^X(A) := P(X \in A)$$

Moreover, P^X is determined by FDD's of X.

Wiener space

for SBM this is then the **Wiener measure**

Cor: SBM is unique when interpreted on the Wiener space

Cor: SBM is locally γ-Hölder a.s. for all $\gamma \in (0, 1/2)$

$A \in \mathcal{B}(C[0,\infty))$

$$\forall t_0 > 0: \sup_{0 \le s < t < t_0} \frac{|B_t - B_s|}{|t-s|^\gamma} < \infty = \{B \in A\}$$

Properties of sample paths

Prop Let $\{B_t : t \in [0,\infty)\}$ be SBM. Then so is $\{W_t : t \in [0,\infty)\}$ for:

1) $W_t := B_{t+a} - B_a$ (for some $a>0$)
2) $W_t := \frac{1}{a} B_{ta^2}$ (for some $a \neq 0$)
3) $W_t := \begin{cases} tB_{1/t} & t>0 \\ 0 & t=0 \end{cases}$ on $\{\lim_{t\to 0^+} W_t = 0\}$ and $W_t := 0$ else

Pf (1), 2) follow from fact that FDD's are characterized by B = Gaussian with $EB_t = 0$, $E(B_tB_s) = t\wedge s$.

For 3) same applies: $E(W_tW_s) \underset{t,s>0}{=} ts\, E(B_{1/t}B_{1/s}) = ts \frac{1}{t} \wedge \frac{1}{s} = t\wedge s$.

NTS: $P(W_t \xrightarrow[t\downarrow 0]{} 0) = 1$.

$$P(W_t \xrightarrow[t\downarrow 0]{} 0) = P\Big(\bigcap_{n\geq 1} \bigcup_{m\geq 1} \bigcap_{k\geq m} \Big\{ \sup_{\substack{t\in \mathbb{Q}_+ \\ 0<t\leq 1/k}} |W_t| < \frac{1}{n} \Big\}\Big)$$

$$= P(\ \ldots\ B_t\ ) = P(B_t \xrightarrow[t\downarrow 0]{} 0) = 1 \quad \boxtimes$$

Corollary: A.s.: $\lim_{t\to\infty} \frac{B_t}{t} = 0.$

Regularity of SBM

Thm (Paley, Zygmund, Wiener '33) Let $\gamma \in (1/2, 1]$.
Then for a.e. sample path of SBM:

$$\forall t \geq 0: \limsup_{s \downarrow t} \frac{|B_t - B_s|}{|t-s|^\gamma} = \infty.$$

In particular, a.e. path is nowhere γ-Hölder and thus nowhere differentiable.