Overcome countability curse

<u>Q:</u> What conditions of FDD's imply existence of continuous version?

<u>Lemma</u> Suppose $\{X_t : t \in \mathbb{R}_+\}$ has continuous paths.
Then $\forall t \in \mathbb{R}_+ \ \forall \varepsilon > 0$: $\lim_{s \to t} P(|X_t - X_s| > \varepsilon) = 0$.

stochastic continuity

<u>Pf</u> Bounded Conv Thm + cont. $\boxtimes$

Stoch. continuity <u>not</u> sufficient to imply continuity

<u>Ex</u> $\{N_t : t \in \mathbb{R}_+\}$ Poisson process,

$$P(|N_t - N_s| > \varepsilon) \underset{\varepsilon > 0}{\leq} P(|N_t - N_s| \geq 1) \leq \underbrace{\sum_{n \geq 1} \frac{|t-s|^n}{n!} e^{-|t-s|}}_{\leq |t-s|}$$

Will moments help?

$$E(|N_t - N_s|^a) \underset{a \in \mathbb{N}}{=} \sum_{n=0}^{\infty} \frac{n^a}{n!} |t-s|^n e^{-|t-s|} \geq |t-s| e^{-|t-s|}$$

Thm (Kolmogorov-Čentsov) Let $\{X_t : t \in \mathbb{R}_+\}$ be $\mathbb{R}$-valued stoch. process s.t.

$$\exists C, a, b > 0 \quad \forall\, 0 \le t \le s: \quad E(|X_t - X_s|^a) \le C|t-s|^{1+b}$$

Then there exists a version $\tilde{X}$ of X s.t.

$$\forall \gamma \in (0, b/a)\ \forall t_0 > 0: \quad \sup_{0 \le s < t \le t_0} \frac{|\tilde{X}_t - \tilde{X}_s|}{|t-s|^\gamma} < \infty$$

In particular, X admits a γ-Hölder cont. version for each $\gamma \in (0, b/a)$.

Pf: Let $\gamma \in (0, b/a)$ and $L \ge 1$ a natural.

Note $\forall\, 0 \le s < t$: $P(|X_t - X_s| > u) \le \frac{1}{u^a} C|t-s|^{1+b}$

So $\forall t \ge 0$: $P(|X_{t+2^{-n}} - X_t| > 2^{-n\gamma}) \le C\, 2^{-n}\, 2^{-n(b-a\gamma)}$

So $P\left(\max_{k=1\cdots L2^n} |X_{k2^{-n}} - X_{(k-1)2^{-n}}| > 2^{-n\gamma}\right) \le \sum_{k=1}^{L2^n} P(|X_{k2^{-n}} - X_{(k-1)2^{-n}}| > 2^{-n\gamma})$

$$\le L2^n \cdot C 2^{-n} 2^{-n(b-a\gamma)} = CL2^{-n(b-a\gamma)}$$

Define: $n_0 := \inf\left\{ m \geq 1 : \left(\forall n \geq m \ \forall k = 1 \ldots L2^{-n} : \left| X_{k2^{-n}} - X_{(k-1)2^{-n}} \right| \leq 2^{-n\gamma} \right) \right\}$

Borel-Cantelli $\Rightarrow$ $P(n_0 < \infty) = 1$.

Assume $n_0 < \infty$ and denote $\mathbb{D}_n := \{ k2^{-n} : k = 0, \ldots, L2^{-n} \}$.

Then $\forall\, s,t \in \mathbb{D}_n$ with $s < t$ $\exists\, s', t' \in \mathbb{D}_{n-1}$:

$$s \leq s' \leq t' \leq t \ \wedge \ s' - s \leq 2^{-n} \ \wedge \ t - t' \leq 2^{-n}$$

Then $$|X_t - X_s| \leq \underbrace{|X_t - X_{t'}| + |X_s - X_{s'}|}_{\leq 2^{-n\gamma} \text{ if } n \geq n_0} + |X_{t'} - X_{s'}|$$

Hence $$\max_{\substack{t,s \in \mathbb{D}_n \\ |t-s| < \delta}} |X_t - X_s| \underset{n \geq n_0}{\leq} 2 \cdot 2^{-n\gamma} + \max_{\substack{t',s' \in \mathbb{D}_{n-1} \\ |t'-s'| < \delta}} |X_{t'} - X_{s'}|$$

Take $\delta < 2^{-n_0+1}$

Now iterate to get

$$\sup_{\substack{t,s \in \bigcup_{k \geq 1} \mathbb{D}_k \\ |t-s| < \delta}} |X_t - X_s| \leq 2 \cdot \sum_{n : 2^{-n+1} > \delta} 2^{-n\gamma} = \frac{2}{1-2^{-\gamma}} \delta^{\gamma}$$

Hence, $$\sup_{\substack{t,s \in \bigcup_{n \geq 1} \mathbb{D}_n \\ 0 \leq t < s \leq L}} \frac{|X_t - X_s|}{|t-s|^{\gamma}} < \infty \quad \text{on } \{n_0 < \infty\}.$$

So X is γ-Hölder on $n_0 < \infty$.

Define: $$\tilde{X}_t := \begin{cases} \lim\limits_{\substack{s \to t \\ s \in \bigcup_{n \geq 1} \mathbb{D}_n}} X_s & \text{on } \{n_0 < \infty\} \\ X_0 & \text{on } \{n_0 = \infty\}. \end{cases}$$

Then $\tilde{X}$ is γ-Hölder. To show that $\tilde{X}$ is a version of X:

$$P(|X_t - X_s| > \varepsilon) \leq \frac{1}{\varepsilon^a} |t-s|^{1+b}$$

Taking limits using Fatou:

$$P(|\tilde{X}_t - X_s| > \varepsilon) \leq \frac{1}{\varepsilon^a} |t-s|^{1+b}.$$ Now take $s \to t$. $\boxtimes$

Thm: A Standard Brownian motion exists

<u>Pf</u> From FDD's & KET: We have prob space $(\Omega, \mathcal{F}, P)$ and $\{B_t : t \in \mathbb{R}_+\}$ with these FDD's.

Since $B_t - B_s = \mathcal{N}(0, |t-s|) = |t-s|^{1/2}\mathcal{N}(0,1)$

we have $E(|B_t - B_s|^a) = |t-s|^{a/2} \underbrace{E(|\mathcal{N}(0,1)|^a)}_{<\infty \ \forall a>0}$

KČT $\Rightarrow$ B admits a γ-Hölder version for each $\gamma \in \left(0, \frac{\frac{a}{2}-1}{a}\right)$ $(a>2)$

This process has same FDD's and continuous paths and so it's <u>a</u> SBM. ⊠

other constructions:
- Levy construction
- Donsker's theorem
- Folland's construction

$\mathcal{B}(\mathbb{R}^{[0,\infty)}) \supsetneq \mathcal{B}(\mathbb{R})^{\otimes[0,\infty)}$