

Proof of KET contin'd $(X, \Sigma) = \text{stand. Borel space}$
 $T \neq \emptyset$ index set

$$\Omega = X^T, \mathcal{F} = \Sigma^{\otimes T}$$

$$\mathcal{A} := \bigcup_{\substack{\text{SET} \\ \text{finite}}} \mathcal{F}_S \quad (\mathcal{F} = \sigma(\mathcal{A}))$$

$$\forall A \in \mathcal{A} \text{ s.t. } A \in \mathcal{F}_S \text{ set } P(A) = \mu_S(A)$$

Lemma $\mathcal{A}$ is an algebra.
 P finitely additive on $\mathcal{A}$.

Prop $\forall \{B_n\}_{n \in \mathbb{N}} \in \mathcal{A}^{\mathbb{N}}: B_n \downarrow \emptyset \Rightarrow P(B_n) \rightarrow 0$.

Lemma: $(X, \Sigma) = \text{standard Borel space}$, $\mu = \text{finite measure on } (X, \Sigma)$.
then $\forall B \in \Sigma: \mu(B) = \sup \{ \mu(C) : C \subseteq B, \text{compact} \}$

Lemma $(X, \Sigma) = \text{standard Borel}$. Then
 $\forall n \geq 1: (X^n, \Sigma^{\otimes n})$ is standard Borel.

Proof of Prop: Suppose $\{B_n\}_{n \in \mathbb{N}} \in \mathcal{A}^{\mathbb{N}}$ s.t. $\varepsilon := \inf P(B_n) > 0$.

WLOG: $\exists \{t_n\}_{n \in \mathbb{N}} \in T^{\mathbb{N}}$ s.t. $\forall n \in \mathbb{N}: B_n \in \mathcal{F}_{\underbrace{t_1, \dots, t_n}_{=: S_n}}$ and $B_n \downarrow$.

Since $(\mathcal{X}, \Sigma^{\mathbb{N}})$ is standard Borel,

$\forall n \in \mathbb{N} \exists C'_n \subseteq \mathcal{X}^n$ compact: $\mu_{S_n}(\pi_{S_n}(B_n) \setminus C'_n) \leq \varepsilon 2^{-n-1}$

Let $C_n := \pi_{S_n}^{-1}(C'_n)$. Then

$$\begin{aligned} P\left(\bigcap_{k=1}^n C_k\right) &\geq P\left(\bigcap_{k=1}^n B_k\right) - \sum_{k=1}^n P(B_k \setminus C_k) \\ &\geq \varepsilon - \sum_{k=1}^n \varepsilon 2^{-k-1} \geq \frac{\varepsilon}{2} \end{aligned}$$

So $\bigcap_{k=1}^n C_k \neq \emptyset$ ($\forall n \in \mathbb{N}$).

Use Cantor diag argument to show $\bigcap_{k \in \mathbb{N}} C_k \neq \emptyset$.

Since $\bigcap_{n \in \mathbb{N}} B_n \supseteq \bigcap_{n \in \mathbb{N}} C_n \Rightarrow B_n \not\downarrow \emptyset$. $\square$

Lemma If claim in Prop holds, then P is cont. add. on $\mathcal{A}$.

Pf Let $\{A_n\}_{n \in \mathbb{N}} \in \mathcal{A}^{\mathbb{N}}$ be disjoint with $A := \bigcup_{n \in \mathbb{N}} A_n \in \mathcal{A}$.

Set $B_n := \bigcup_{k > n} A_k$. Then $A = A_1 \cup \dots \cup A_n \cup B_n$.

So $B_n \in \mathcal{A}$ and by finite add:

$$P(A) = \sum_{k=1}^n P(A_k) + P(B_n)$$

Since $B_n \downarrow \emptyset$ we have $P(B_n) \rightarrow 0$ and $P(A) = \sum_{n \in \mathbb{N}} P(A_n)$ $\square$

Proof of KET Hahn-Kolmogorov:

P extends to $\mathcal{F} = \sigma(\mathcal{A})$.

a measure on

So $(\Omega, \mathcal{F}, P)$ is prob. space.

For $\omega = (w_t)_{t \in T} \in \Omega$, set $X_t(\omega) := w_t$.

Then $\forall n \geq 1 \forall t_1, \dots, t_n \in T \forall A \in \sum_{i=1}^n \mathcal{G}_{t_i}$: $P((X_{t_1}, \dots, X_{t_n}) \in A) = P(\pi_{(t_1, \dots, t_n)}^{-1}(A))$
 $= \mu_{(t_1, \dots, t_n)}(A)$ $\square$

Uniqueness π - λ -Theorem

Def Given collection $\mathcal{P}, \mathcal{L}$ of subsets of Ω , we say:

• $\mathcal{P}$ is π -system if $\forall A, B \in \mathcal{P}: A \cap B \in \mathcal{P}$.

• $\mathcal{L}$ is λ -system if

– $\emptyset, \Omega \in \mathcal{L}$

– $\forall A, B \in \mathcal{L}: A \subseteq B \Rightarrow B \setminus A \in \mathcal{L}$

– $\forall \{A_n\}_{n \in \mathbb{N}} \in \mathcal{L}^{\mathbb{N}}: A_n \uparrow A \Rightarrow A \in \mathcal{L}$.

Ex • Any σ -algebra

• Given two prob. measures on (Ω, Σ) :

$$\mathcal{L} := \{A \in \Sigma: \mu(A) = \nu(A)\}$$

Thm (Dynkin π - λ thm) If $\mathcal{P} = \pi$ -system, $\mathcal{L} = \lambda$ -system,
 $\mathcal{P} \subseteq \mathcal{L} \Rightarrow \sigma(\mathcal{P}) \subseteq \mathcal{L}$.

Since $\mathcal{A}$ is π -system, Dynkin's thm shows that the extension to $(\Omega, \mathcal{F})$ is unique.

Note $T = [0, \infty)$, set $U := \exp(1) \perp \{X_t : t \in [0, \infty)\}$

$$\text{Set } Y_t := \begin{cases} X_{t+1} & t = U \\ X_t & t \neq U \end{cases}$$

Then Y has same FDD's as X !

Recall: $\sigma(X_t : t \in T) = \sigma(\{ \{X_t \in A\} : t \in T, A \in \Sigma \})$

Lemma $\sigma(X_t : t \in T) = \bigcup_{\substack{S \subseteq T \\ \text{Countable}}} \sigma(X_t : t \in S)$

Corollary: Let $\{X_t : t \in [0, \infty)\}$ be $\mathbb{R}$ -valued process realized via coordinate projections on $(\mathbb{R}^{[0, \infty)}, \mathcal{B}(\mathbb{R})^{\otimes [0, \infty)})$

The $\{t \mapsto X_t \text{ continuous}\} \notin \sigma(X_t : t \in [0, \infty))$

Moreover, $\forall A \in \sigma(X_t : t \in [0, \infty))$:

$$A \subseteq \{t \mapsto X_t \text{ cont}\} \Rightarrow A = \emptyset.$$

How to proceed? Will work with:

Def Given a process $\{X_t : t \in T\}$ on $(\Omega, \mathcal{F}, P)$, its modification/version is any process $\{Y_t : t \in T\}$ on $(\Omega, \mathcal{F}, P)$ s.t.

$$\forall t \in T: P(X_t = Y_t) = 1.$$