Time change: $\begin{cases} du = Z_t^2\,dt\,, & d\tilde{B}_u = Z_t\,dB_t \\ \int_0^{t(u)} Y_s Z_s\,dB_s = \int_0^u Y_{t(v)}\,d\hat{B}_v \end{cases}$

Thm Let $\sigma : \mathbb{R} \to \mathbb{R}$ be s.t. $\sigma^{-2} \in L^{1,loc}(\mathbb{R})$. Let $W = $ SBM with $W_0 = x$ a.s. Then

$$\forall t \geq 0 : U_t := \int_0^t \frac{1}{\sigma(W_s)^2}\,ds < \infty \text{ a.s.} \wedge U_t \xrightarrow[t\to\infty]{} \infty \text{ a.s.}$$

So for $T(u) := \inf\{t \geq 0 : U_t \geq u\}$,

$$B_u := \int_0^{T(u)} \frac{1}{\sigma(W_s)}\,dW_s \quad \text{is SBM started at } 0.$$

and

$$X_u := W_{T(u)}$$

solves the SDE $dX_u = \sigma(X_u)\,dB_u$ with initial data $X_0 = x$

Pf Pick $M > 0$, let $\hat{\tau}_M := \inf\{t \geq 0 : |W_t - x| \geq M\}$.

Then ...

$$E\left(\mathbb{1}_{\hat{\tau}_M > t} \int_0^t \frac{1}{\sigma(W_s)^2} ds \right) = \int_0^t E\left(\frac{1}{\sigma(W_s)^2} \mathbb{1}_{\hat{\tau}_M > t} \right) ds$$

$$\leq \left(\int_0^t \frac{1}{\sqrt{s}} ds \right) \int_{-M}^{M} \frac{1}{\sigma(x+y)^2} dy < \infty$$

So $U_t < \infty$ a.s. on $\bigcup_{M \geq 0} \{ \hat{\tau}_M > t \}$, which is full measure.

Next: $U_\infty := \lim_{t \to \infty} U_t$ exists in $[0, +\infty]$. and

by recurrence and SMP $\quad U_\infty = \int_0^\infty \frac{1}{\sigma(W_s)^2} ds$

So U_∞ is bounded by sum of iid copies of $\int_0^{\hat{\tau}_x} \frac{1}{\sigma(W_s)^2} ds$

where $\hat{\tau}_x := \inf\{ t \geq 1 : W_t = x \}$. So $U_\infty = \infty$ a.s.

Hence, $B.$ is SBM. by Lévy characterization

Change of var:

$$\int_0^u \sigma(W_{T(u)}) dB_u = \int_0^{T(u)} \cancel{\sigma(W_t)} \frac{1}{\cancel{\sigma(W_t)}} dW_t = W_{T(u)} - W_{T(0)}$$

So $X_u := W_{T(u)}$ solves $dX_u = \sigma(X_u) dB_u$ with $X_0 = x$. $\boxtimes$

$$dX_t = \sigma(X_t)\,dB_t$$

Uniqueness is linked to 0's of σ. Two issues.

1) can I reach a zero point in finite time?

2) if reached, can I get out?

Answered completely in:

<u>Thm</u> (Engelbert–Schmidt) Let $\sigma : \mathbb{R} \to \mathbb{R}$ be measurable. Set

$$I(\sigma) := \bigcap_{\varepsilon > 0} \left\{ x \in \mathbb{R} : \int_{(-\varepsilon,\varepsilon)} \frac{1}{\sigma(x+y)^2}\,dy = +\infty \right\}$$

$$Z(\sigma) := \{ x \in \mathbb{R} : \sigma(x) = 0 \}$$

Then $dX_t = \sigma(X_t)\,dB_t$ has weak existence and uniqueness in law if and only if $I(\sigma) = Z(\sigma)$.

Girsanov Thm

Idea: Drift term effectively changes the mean.
There is a way to change mean by "tilting" the measure.

Lemma Let $X_1, \dots, X_n$ be independent with exp. moments.
For $\lambda_1, \dots, \lambda_n \in \mathbb{R}$, let $P_\lambda(A) := E\left(1_A \exp\left\{\sum_{i=1}^n [\lambda_i X_i - \log E e^{\lambda_i X_i}]\right\}\right)$
Then P_λ is a probability measure, $X_1 \dots X_n$ are still independent under P_λ, yet $E_\lambda(X_i) = \frac{\varphi_i'(\lambda_i)}{\varphi_i(\lambda)}$, where $\varphi_i(\lambda) := E(e^{-\lambda X_i})$

Pf Suffices to show this for $n=1$.

$$E\left(e^{\lambda X - \log\varphi(\lambda)}\right) = \frac{1}{\varphi(\lambda)}\varphi(\lambda) = 1$$

$$E\left(X e^{\lambda X - \log\varphi(\lambda)}\right) = \frac{1}{\varphi(\lambda)} \frac{d}{d\lambda} E\left(e^{\lambda X}\right) = \frac{\varphi'(\lambda)}{\varphi(\lambda)}$$

Assume $\varphi(\cdot) < \infty$ on neighbourhood of λ. $\boxtimes$

For $X = \mathcal{N}(\mu, \sigma^2)$, $\log \varphi(\lambda) = \lambda\mu + \frac{\lambda^2}{2}\sigma^2$.

Thm (Girsanov) Let $B = \text{SBM}$, $\{\mathcal{F}_t\}_{t\geq 0}$ = Br. filtration, $Y \in \mathcal{V}^{loc}$, $t > 0$. Set

$$M_t := \exp\left\{ \int_0^t Y_s \, dB_s - \frac{1}{2}\int_0^t Y_s^2 \, ds \right\}.$$

Assume $EM_t = 1$. Then $\tilde{P}(A) = E(1_A M_t)$, $A \in \mathcal{F}_t$, is prob. measure on $(\Omega, \mathcal{F}_t)$ and

$$\left\{ B_s - \int_0^t Y_s \, ds : s \in [0,t] \right\}$$

is SBM under $\tilde{P}$.

Pf claim: Under $EM_t = 1$, $\{M_{t\wedge s} : s \geq 0\}$ is a martingale.

Pf Itô: M is local martingale, so setting

$$\tau_n := \inf\left\{ u \geq 0 : \left|\int_0^u Y_s \, dB_s\right| \geq n \right\}$$

$\{M_{s\wedge t\wedge \tau_n} : s \geq 0\}$ is bounded loc. martingale and so a martingale. So

$$E(M_{t\wedge\tau_n} \mid \mathcal{F}_s) = M_{t\wedge s\wedge\tau_n}, \quad E(\cdots) = 1$$

So Fatou $E(M_{t\wedge s}) \le \liminf_{n\to\infty} E(M_{t\wedge s\wedge \tau_n}) = 1$ ← by

Conditional Fatou:

$$E(M_t \mid \mathcal{F}_s) \le M_{t\wedge s} \quad \text{and} \quad E(M_{t\wedge s}) \le 1$$

But $E(M_t) = 1$ implies $=$ must hold a.s.

So $\{M_{t\wedge s} : s \ge 0\}$ is a martingale.

2) $\tilde{B}_s := B_s - \int_0^s Y_u \, du$, take Z bounded adapted process

$$N_s := \exp\left\{ i \int_0^s Z_s \, d\tilde{B}_s + \int_0^s Z_s^2 \, ds \right\}$$

Itô: $\{M_{t\wedge s} N_{t\wedge s} : s \ge 0\}$ is martingale.

So for $Z_s := \sum_{j=1}^n \lambda_j \mathbb{1}_{(t_{j-1}, t_j]}(s)$ we get

$$1 = E(M_0 N_0) = E(M_t N_t) = \tilde{E}\left(\exp\left\{ i \sum_{j=1}^n \lambda_j (\tilde{B}_{t_j} - \tilde{B}_{t_{j-1}}) \right\} \right) \exp\left\{ +\frac{1}{2} \sum_{j=1}^n \lambda_j^2 (t_j - t_{j-1}) \right\}$$

So $\hat{B}$ is SBM. $\boxtimes$