Weak solutions and all that

Tanaka — absence of strong solution — also
$dX_t = \mathrm{sgn}(X_t)\,dB_t$ — 〃 — pathwise uniqueness

Def (Uniqueness in law) We say and SDE has *uniqueness in law* if any two weak solutions X an $\tilde{X}$ (different prob. spaces etc allowed) obey:

$$X_0 \overset{\text{law}}{=} \tilde{X}_0 \implies \{X_t : t \geq 0\} \overset{\text{law}}{=} \{\tilde{X}_t : t \geq 0\}$$

☺ Tanaka eg. (all weak solutions are SBM)

☹ $dX_t = \mathbb{1}_{\mathbb{R}\setminus\{0\}}(X_t)\,dB_t$ $\quad \begin{cases} X_t = B_t \\ X_t = 0 \end{cases}$ for $X_0 = 0$

Q: which is stronger?
Pathwise on in-law uniqueness?

Abstract nonsense theory of Yoshida & Watanabe

Key idea: Uniformizing map

canonical space

$$\mathcal{X} = \mathbb{R} \times C[0,\infty) \times C[0,\infty)$$

$$\phi(\omega) := (X_0(\omega), B(\omega), X(\omega))$$

maps $\Omega \to \mathcal{X}$.

Fact 1 ϕ maps any weak solution to a strong solution on $\mathcal{X}$.

Fact 2 Pathwise uniqueness $\Rightarrow$ uniqueness in law

Fact 3 existence of solution map

Thm: Suppose an SDE admits a weak solution for all initial data and has pathwise uniqueness.

Then $\exists\, \eta : \mathbb{R} \times C[0,\infty) \to C[0,\infty)$ s.t.

for any prob. space $(\Omega, \mathcal{F}, P)$ with SBM B, Br. filtration $\{\mathcal{F}_t\}$:

$$X := \eta(X_0, B)$$

is the strong solution of the SDE with initial data X_0.

Methods for solving SDEs

1) Reduction to an ODE

Lemma Let X be a solution to SDE

$$dX_t = a(t, X_t)\,dt + dB_t$$

Then $Y_t := X_t - B_t$ solves the ODE

$$\frac{dY_t}{dt} = a(t, Y_t + B_t)$$

Pf X solves SDE :⟺

$$X_t = X_0 + \int_0^t a(s, X_s)\,ds + B_t$$

So Y obeys

$$Y_t = Y_0 + \int_0^t a(s, Y_s + B_s)\,ds$$

so it solves ODE in Lebesgue sense. $\boxtimes$

Ex Langevin eq: $dX_t = -\nabla H(X_t)\,dt + dB_t$

$\mu(dx) = e^{-H(x)}\,dx$

equilibrium measure

2) <u>coordinate change</u>

$$dX_t = a(X_t)\,dt + \sigma(X_t)\,dB_t, \quad f \in C^2(\mathbb{R})$$

$$df(X_t) = \Big[\underbrace{f'(X_t)a(X_t) + \tfrac{1}{2}f''(X_t)\sigma(X_t)^2}_{\overset{\text{WANT}}{=}\,0}\Big]dt + f'(X_t)\sigma(X_t)\,dB_t$$

when $\frac{1}{2}f''\sigma^2 + f'a = 0$

$$f'(x) = \exp\left\{-2\int_{x_0}^{x} \frac{a(u)}{\sigma(u)^2}\,du\right\}$$

<u>Lemma</u> Suppose $\exists\ \alpha < \beta$ s.t. $x \longmapsto \frac{a(x)}{\sigma(x)^2} \in L^{1,loc}((\alpha,\beta))$ and, for some $x_0 \in (\alpha,\beta)$,

$$f(x) := \int_{x_0}^{x} \exp\left\{-2\int_{x_0}^{v} \frac{a(u)}{\sigma(u)^2}\,du\right\} dv$$

obeys $f(x) \xrightarrow[x\downarrow\alpha]{} -\infty$, $f(x) \xrightarrow[x\uparrow\beta]{} +\infty$. Then if X solves above SDE then $Z_t := f(X_t)$ solves $dZ_t = \tilde{\sigma}(Z_t)\,dB_t$ for

$$\tilde{\sigma}(z) := f' \circ f^{-1}(z)\ \sigma \circ f^{-1}(z),$$ and vice versa.

3) Time change

So far, we reduced to SDE of form $dX_t = \sigma(X_t)\,dB_t$.

Idea: Time change this to make RHS $d\tilde{B}_u$.

Thm (Time change for Itô integral)

Let $B = $ SBM, $\{\mathcal{F}_t\}_{t\geq 0} = $ Brownian filtration, $\mathcal{F}_0$ contains P-null sets,

Suppose $Z \in \mathcal{V}^{loc}$ is s.t.

$$U_t := \int_0^t Z_s^2\,ds \xrightarrow[t\to\infty]{} \infty \quad \text{a.s.}$$

Define $T(u) := \inf\{t \geq 0 : U_t \geq u\}$, and recall that $\tilde{B}_u := \int_0^{T(u)} Z_s\,dB_s$ is SBM w.r.t. $\{\mathcal{F}_{T(u)}\}_{u\geq 0}$.

Given a LC process $\{Y_t : t\geq 0\}$ adapted to $\{\mathcal{F}_t\}_{t\geq 0}$ s.t. $YZ \in \mathcal{V}^{loc}$, $\{Y_{T(u)} : u\geq 0\}$ is adapted to $\{\mathcal{F}_{T(u)}\}_{u\geq 0}$ and LC and

$$\forall u\geq 0: \begin{cases} \displaystyle\int_0^u Y_{T(v)}^2\,dv = \int_0^{T(u)} Y_s^2 Z_s^2\,ds \\ \displaystyle\int_0^u Y_{T(v)}\,d\tilde{B}_v = \int_0^{T(u)} Y_s Z_s\,dB_s \quad \text{a.s.} \end{cases}$$

Pf L^2-limit + localization $\Rightarrow$ suffices to prove this for $T(s) \geq a \iff U_a < s$
simple processes. For $Y_s = 1_{(a,b]}(s)$ we get

$$\int_0^u Y_{T(v)}^2 \, dv = \lambda\left(\{v \in \mathbb{R}_+ : v < u \wedge T(v) \in (a,b]\}\right)$$

$$= U_{b \wedge u} - U_{a \wedge u}$$

$$= \int_0^{b \wedge T(u)} Z_s^2 \, ds - \int_0^{a \wedge T(u)} Z_s^2 \, ds = \int_0^{T(u)} \overset{Y_s^2}{\overset{\|}{1_{(a,b]}(s)}} Z_s^2 \, ds$$

For stoch. integral assume $Y_s = 1_{(0,a]}(s)$

Then $\int_0^u Y_{T(v)} \, d\hat{B}_v = \int_0^u 1_{(0,U_a]}(s) \, d\hat{B}_s$

$$= \bar{B}_{U_a \wedge u} = \int_0^{T(u) \wedge a} Z_s \, dB_s \quad \boxtimes$$

Calculus Rules: $du = Z_t^2 \, dt$, $d\tilde{B}_u = Z_t \, dB_t$

$$\int_0^{t(u)} Z_s^2 \, ds = u \Rightarrow \int_0^{t(u)} Y_s Z_s \, dB_s =: \int_0^u Y_{t(v)} \, d\bar{B}_v$$