Tanaka equation

<u>recall</u> Under locally uniform Lipschitz cont. of coefficients we have existence and <u>pathwise uniqueness</u> of solution to SDE.

$$\hookrightarrow P(X_0=\tilde{X}_0)=1 \Rightarrow P(\forall t\geq 0: X_t=\tilde{X}_t)=1$$

Today: an example where this fails.

$$\mathcal{F}_t^B=\sigma(B_s : s\leq t)$$
$$\tilde{\mathcal{F}}_t^B=\sigma(\mathcal{N}\cup\mathcal{F}_t^B)$$

<u>Prop</u> Let B be SBM, $\{\tilde{\mathcal{F}}_t^B\}$ = augmented filtration. Then

$$dX_t = \mathrm{sgn}(X_t)\,dB_t$$

where $\mathrm{sgn}(x) := \begin{cases} 1 & , x\geq 0 \\ -1 & , x<0 \end{cases}$

admits no strong solutions.

$X_0>0 \ldots X_t = X_0 + B_t$
until $\tau_0 = \inf\{s\geq 0 : X_0+B_s=0\}$
after τ_0, X follows the excursion if $X\geq 0$ but opposite when $X<0$. The latter runs into a conflict with SDE.

Fact: Itô formula applies to f_ε.

$$f_\varepsilon(B_t) = f_\varepsilon(B_0) + \int_0^t \operatorname{sgn}(B_s)\,dB_s + \frac{1}{2\varepsilon}\int_0^t \mathbb{1}_{\{|B_s|<\varepsilon\}}\,ds$$

Note: $|x| \le f_\varepsilon(x) \le |x| \vee \varepsilon$ so $f_\varepsilon(B_t) \xrightarrow[\varepsilon\downarrow 0]{} |B_t|$

$$|f_\varepsilon'(x) - \operatorname{sgn}(x)| \le \mathbb{1}_{(-\varepsilon,\varepsilon)}(x)$$

So
$$E\left(\left|\int_0^t f_\varepsilon'(B_s)\,dB_s - \int_0^t \operatorname{sgn}(B_s)\,dB_s\right|^2\right)$$
$$\overset{\text{Itô}}{\le} E\int_0^t \mathbb{1}_{|B_s|<\varepsilon}\,ds = \int_0^t P(|B_s|<\varepsilon)\,ds$$
$$\le 2\varepsilon\int_0^t \frac{1}{\sqrt{s}}\,ds = 4\varepsilon\sqrt{t} \qquad \boxed{X}$$

Pf of Prop: Assume X is strong solution to $dX_t = \operatorname{sgn}(X_t)\,dB_t$ adapted to $\{\tilde{\mathcal{F}}_t^B\}_{t\ge 0}$. Then

$$\langle X\rangle_t = \int_0^t \operatorname{sgn}(X_s)^2\,ds = \int_0^t ds = t$$

Levy char: X is SBM. Then

$$\int_0^t \operatorname{sgn}(X_s)\,dX_s = \int_0^t \operatorname{sgn}(X_s)\operatorname{sgn}(X_s)\,dB_s = \int_0^t dB_s = B_t$$

Taking $\varepsilon_n := \frac{1}{n^2}$, Tanaka's formula gives

$$B_t = \int_0^t \operatorname{sgn}(X_s)\, dX_s = |X_t| - |X_0| - \lim_{n\to\infty} \frac{1}{2\varepsilon_n} \int_0^t 1_{|X_s|<\varepsilon}\, ds \quad \text{a.s.}$$

Hence we get:

$$\forall t \geq 0: \quad \mathcal{F}_t^B \subseteq \tilde{\mathcal{F}}_t^{|X|}$$

But X is a solution, so $\mathcal{F}_t^X \subseteq \mathcal{F}_t^B$ and so

$$\forall t \geq 0: \quad \tilde{\mathcal{F}}_t^X \subseteq \tilde{\mathcal{F}}_t^B \subseteq \tilde{\mathcal{F}}_t^{|X|}$$

<u>Lemma</u> $\forall t > 0: \{X_t - X_0 > 0\} \notin \tilde{\mathcal{F}}_t^{|X|}$. (wlog $X_0 = 0$)

Pf Let $\mathcal{G}_t := \{A \in \mathcal{F} : E(X_t 1_A) = 0\}$.

Then $\mathcal{G}_t$ is σ-algebra. Now for any $B \in \mathcal{B}(\mathbb{R})$, $s \leq t$:

$$E(X_t 1_{|X_s| \in B}) = \underbrace{E((X_t - X_s) 1_{|X_s| \in B})}_{=0} + \underbrace{E(X_s 1_{|X_s| \in B})}_{=0}$$

So $\{|X_s| \in B\} \in \mathcal{G}_t$. So $\mathcal{F}_t^{|X|} \subseteq \mathcal{G}_t$. Since $\mathcal{N} \subseteq \mathcal{G}_t$, we have $\tilde{\mathcal{F}}_t^{|X|} \subseteq \mathcal{G}_t$.

But $E(X_t 1_{\{X_t > 0\}}) > 0$ unless $t = 0$. So $\{X_t > 0\} \notin \tilde{\mathcal{F}}_t^{|X|}$. $\boxtimes$

Note $dX_t = \operatorname{sign}(X_t)\, dB_t$ where $\operatorname{sign}(x) := \begin{cases} +1 & x > \\ -1 & x < \\ 0 & x \end{cases}$

has strong solution $X_t = X_0 + \operatorname{sign}(X_0)^{\dagger} B_{t \wedge \tau_0}$.

Note There is a way so solve Tanaka eq:

Let $X = $ SBM. Define $B_t := \int_0^t \operatorname{sgn}(X_s)\, dX_s$

defines SBM s.t. $dX_t = \operatorname{sgn}(X_t)\, dB_t$.

This produces an instance of

Def A weak solution to an SDE is the four-tuple

$$\big((\Omega, \mathcal{F}, P), \{\mathcal{F}_t\}_{t \geq 0}, B, X\big)$$

s.t. X is a strong solution to the SDE for

standard setting provided by $\big((\Omega, \mathcal{F}, P), \{\mathcal{F}_t\}, B, X_0\big)$.

Note All we needed was to change filtration to $\{\mathcal{F}_t^X\}$.

Weaking of concept of solution may ruin uniqueness.

Lemma Let X be a strong solution to $dX_t = \operatorname{sgn}(X_t)\, dB_t$ for a fourtuple above.

Set $\tau := \inf\{t \geq 1 : X_t = 0\}$

and define $\tilde{X}_t := \begin{cases} X_t & t \leq \tau \\ -X_t & t > \tau. \end{cases}$

Then $\tilde{X}$ is also a strong solution ?!.

So no pathwise uniqueness for Tanaka equation.

Pf $\tilde{X}$ is continuous, adapted because

$$\{\tilde{X}_t \in A\} = \{X_t \in A\} \cap \{\tau \leq t\} \cup \left(\{-X_t \in A\} \cap \{\tau > t\}\right)$$

SMP / Levy char: $\tilde{X}$ is SBM

$$\int_0^t \operatorname{sgn}(\tilde{X}_s)\, dB_s = \int_0^t \left[2 \cdot \mathbb{1}_{\tau > s} - 1\right] \operatorname{sgn}(X_s)\, dB_s$$

$$= 2 \cdot (X_{\tau \wedge t} - X_0) - (X_t - X_0)$$

$$= 2 X_{t \wedge \tau} - X_t - X_0 = \tilde{X}_t - \tilde{X}_0 . \quad \boxtimes$$